

## Homework 8 – PROBABILISTIC METHODS

Janos Pach &amp; Nabil Mustafa

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## Questions

1. Let  $G = (V, E)$  be a graph with  $2n$  vertices and  $m$  edges. Prove that there exists a cut with  $n$  vertices on each side, and with at least  $\frac{mn}{2n-1}$  edges across the cut.

**Solution:** Do the counting proof (as done in class), but do not count over all  $2^{n-1}$  cuts, but only cuts with  $n$  vertices on each side (there are  $\binom{2n}{n}$  such cuts).

2. Given a graph  $G = (V, E)$  with  $n$  vertices, consider the following method for constructing a cut of  $G$ . Let  $V = \{v_1, \dots, v_n\}$  be the vertices of  $G$ . Start with the cut consisting of vertices  $v_1$  and  $v_2$  on two different sides, i.e., the cut  $(\{v_1\}, \{v_2\})$ . Now add each of the remaining vertices  $v_3, \dots, v_n$  to this cut one-by-one as follows: assume we have already added the vertices  $v_3, \dots, v_{i-1}$  to the cut. Then add the next vertex  $v_i$  to the side of the cut to which  $v_i$  has fewer edges. Prove that the final cut has at least  $m/2$  edges across it.

**Solution:** There are  $n-2$  cuts constructed along the way, as we add each vertex  $v_i$ ,  $i = 3 \dots n$ . Argue, by induction, that at each of those cuts, at least half of the currently present edges are going across the cut. When the next vertex is added, by the way we place the new vertex, it has at least half its edges going across the cut. Use this fact, together with induction, for the proof.

3. Consider the following deterministic way of constructing  $\epsilon$ -nets: pick an element of  $X$  that hits maximum number of sets. Add this element to our  $\epsilon$ -net, and inductively compute an  $\epsilon$ -net for the remaining sets that were not hit. Show that this constructs an  $\epsilon$ -net of size  $2k \log m$ . Recall that each set has size at least  $n/k$ .

**Solution:** Lets say the sets are  $\mathcal{R} = \{P_1, \dots, P_m\}$ . Pick a point  $p \in X$  that hits the largest number of sets of  $\mathcal{R}$ , add  $p$  to our  $\epsilon$ -net, remove the sets hit by  $p$  from  $\mathcal{R}$ , and repeat. Since each set contains a large number of points, there has to be a point hitting several sets. Specifically, by pigeonhole principle, there exists a point hitting  $\sum |P_i|/n \geq m/k$  sets. There are at most  $m - m/k = m(1 - 1/k)$  remaining sets of  $\mathcal{R}$ . By repeating the argument, there exists a point hitting at least  $m(1 - 1/k)1/k$  of these remaining sets. After adding the second point, and further removing the hit sets, we have at most  $m(1 - 1/k) - m(1 - 1/k)1/k = m(1 - 1/k)^2$  sets remaining. In general, at the  $i$ -th iteration, we have added  $i$  points to our  $\epsilon$ -net, and there are  $m(1 - 1/k)^i \leq me^{-i/k}$  unhit sets remaining. For  $i = O(k \ln m)$ , this becomes some constant, after which one can just add one point from each remaining set to the  $\epsilon$ -net. Therefore, we can hit all sets with  $O(k \ln m)$  points.

4. Pick  $k$  random numbers from the set  $\{1, \dots, n\}$  (a number may be picked multiple times). Show that the expected value of the minimum number picked is approximately  $\frac{n}{k+1}$ . You may use the fact that  $\sum_{i=1}^n i^m$  is approximately  $n^{m+1}/(m+1)$ .

**Solution:** Use linearity of expectation.  $X_i$  is 1 if  $i$  lies below the minimum number picked. Then the required quantity is  $E[X] = \sum_i E[X_i]$ .  $E[X_i] = \text{prob}[\text{all the numbers picked are above } i] = ((n-i)/n)^k$ . So expected is  $\sum_i (i/n)^k$  which is approximately  $n/k+1$ .

**Bonus Problem.** Let  $\mathcal{T}$  be the family of all nonempty subsets of  $\{1, 2, \dots, n\}$  with the property that any  $T \in \mathcal{T}$  contains no two consecutive integers. For every  $T \in \mathcal{T}$ , let  $p_T$  denote the product of the squares of all elements of  $T$ . Prove that the sum of the numbers  $p_T$  over all elements  $T \in \mathcal{T}$  is  $(n+1)! - 1$ .

10 points.