Introduction to Combinatorics Spring, 2011

Homework 7 – Probabilistic Methods

Janos Pach & Nabil Mustafa *6 April*

Questions

1. We toss a fair coin n times. What is the expected number of 'runs'? Runs are consecutive tosses with the same result. For instance, the toss sequence HHHTTHTH has 5 runs.

   Solution: Let X_i be 1 if a runs starts at i-th coin toss. Then the required quantity is $\sum_i X_i$. $X_1 = 1$, and $X_i = 1/2$ for $i = 2 \ldots n$.

2. For a permutation π, let $f(\pi)$ be the number of fixed points of π. What is $E[f(\pi)]$ for a random permutation π on n elements.

   Solution: Let X_i be 1 if the i-th position is a fixed point. Then the expected number of fixed points in a random permutation are simply $\sum_i E[X_i]$. And $E[X_i] = 1/n$.

3. The number of *left maxima* for a permutation π of $\{1, \ldots, n\}$ is defined to be the number of indices $i \in [n]$ such that $\pi(i) > \pi(j)$ for all $j < i$. Using linearity of expectation, compute the expected number of left maxima for a random permutation?

   Solution: Let X_i be 1 if the i-th element is a left maxima. This happens if the i-the element is the largest of the elements $1 \ldots i$, and in a random permutation the probability of that is exactly $1/i$. Therefore, the expected number of left maxima is $\sum E[X_i] = \sum 1/i$, which is the harmonic series.

4. Let X be a set of n elements, and $\mathcal{M}$ a set system on X, i.e., $\mathcal{M} = \{S_1, \ldots, S_m\}$, where $S_i \subseteq X$ and $|S_i| = k$ for all $i = 1 \ldots m$. Prove that if $m < 2^{k-1}$, then X can be two-colored (i.e., each element of X can be colored either 'red' or 'blue') such that no set S_i is monochromatic (a set S is monochromatic if all the elements in S have the same color).

   Solution: Each element gets color red with probability 1/2 and color blue with probability 1/2. Then the probability that a fixed set gets all blue or all red vertices is at most $2 \cdot 1/2^k$. By the union bound, the probability that at least one set gets all red or all blue vertices is at most $m \cdot 1/2^{k-1}$. If $m < 2^{k-1}$, then this quantity is less than 1, and so there exists the required coloring.

5. Can you construct a tournament T on 6 vertices such that for *any* pair of vertices $u, v \in T$, there is a third vertex w such that w beats both u and v? What about a tournament with 7 vertices?

   Solution: By trying a few examples, one can construct such a tournament on 7 vertices. For 6 vertices, there is no such tournament. This can be shown by assuming the existence of such a tournament, and deriving a contradiction by considering pairs of vertices which have a common incoming neighbor (one will eventually reach a situation where it is impossible for a pair to have a common neighbor).

Bonus Problem. Prove that there exist four positive integers a_1, a_2, a_3, a_4 such that for any integer $w \in \{1, \ldots, 40\}$, there exist $c_i \in \{-1, 0, +1\}$, $i = 1, \ldots, 4$, such that $w = c_1 \cdot a_1 + c_2 \cdot a_2 + c_3 \cdot a_3 + c_4 \cdot a_4$.

10 points.