Introduction to Combinatorics Spring, 2011

Homework 6 – Dilworth's theorem

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Questions

1. Prove the following: Let k, l be natural numbers. Then every sequence of real numbers of length $kl+1$ contains a non-decreasing subsequence of length $k+1$ or a decreasing subsequence of length $l+1$.

   Solution: Exactly the same proof as for increasing/decreasing sequences done in class. For each number in the sequence, consider the longest increasing sequence ending at that number. Assuming no non-decreasing subsequence of length $k+1$, that will be an integer from 1 to k. Now, the indices with the same-length longest-increasing sequence form a decreasing sequence, and by pigeonhole, that must be of length at least $l+1$.

2. Let S be a sequence of n (not necessarily distinct) integers. Assume $n > rst$, where r, s, t are three positive integers. Then prove that either there exists a strictly increasing subsequence of size r, or a strictly decreasing subsequence of size s or a subsequence of size t consisting of the same integer.

   Solution: Remove all copies (except one) of any duplicate numbers. Then if no number was repeated more than t times, the remaining sequence has length greater than rs. Now find the longest increasing or decreasing subsequence here.

3. Given a set I of n intervals in $\mathbb{R}$, assume that there is no 'nested' set of intervals with size k (a set of intervals are nested if for every pair, one is completely contained inside the other). Then prove that there exists a subset of size n/k where no pair of intervals are nested.

   Solution: The 'nesting' property defines a partial order. By Dilworth's theorem, if the longest chain has size k, the set of intervals can be partitioned into k sets where each set is an anti-chain. And one such anti-chain has size at least n/k.

4. Given a set I of n intervals in $\mathbb{R}$, prove that either one can find $\sqrt{n}$ disjoint intervals in I, or $\sqrt{n}$ intervals $I' \subseteq I$ where all the intervals in I' contain a common point.

   Solution: Form the partial order where $I_i \preceq I_j$ iff I_i is disjoint, and to the left of I_j. Either this has a chain of length $\sqrt{n}$, and these are the required disjoint intervals. Or an anti-chain of same size, which form the intervals with a common intersection point.

Bonus Problem. Alice and Bob play a game where they have to write bits, which can be either 0 or 1, one after the other on a piece of paper. Alice will write the first bit, Bob writes the next bit, then Alice, then Bob and so on. The player who writes the bit after which there are two repeated sequences (can be disjoint, or overlapping) of length n loses. Show that the game always ends, and that for odd n, Bob always wins.

10 points.