Introduction to Combinatorics Spring, 2011

Homework 5 – Sperner's Theorem and Unit Distances

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Questions

1. Given a set system $\mathcal{F}$ over the base set $\{1, \ldots, n\}$, we call $\mathcal{F}$ *semi-independent* if it contains no three sets A, B, C such that $A \subset B \subset C$. Prove that $|\mathcal{F}| \leq 2\binom{n}{\lfloor n/2 \rfloor}$.

   Solution: Exactly the same proof as for Sperner's theorem, except that each permutation could contain two sets of $\mathcal{F}$.

2. Let $a_1, \ldots, a_n$ be real numbers with $|a_i| \geq 1$. Let $p(a_1, \ldots, a_n)$ be the number of vectors $(\epsilon_1, \ldots, \epsilon_n)$, where $\epsilon_i = \pm 1$, such that

   $$-1 < \sum_{i=1}^{n} \epsilon_i a_i < 1$$

   Prove that for any $a_1, \ldots, a_n$, we have $p(a_1, \ldots, a_n) \leq \binom{n}{\lfloor n/2 \rfloor}$.

   Solution: Apply Sperner's theorem by representing each set with its characteristic vector.

3. Let X be an n-element set, and let $S_1, \ldots, S_n$ be subsets of X such that $|S_i \cap S_j| \leq 1$ for all $1 \leq i < j \leq n$. Prove that at least one set has size at most $C\sqrt{n}$ for some absolute constant C.

   Solution: Construct a bi-partite graph where each element of X is represented by a vertex, and each set S_j is represented by the vertex s_j. Then add all edges between the vertex s_j, and all the vertices in X that are contained in S_j. This graph is $K_{2,2}$-free, and so has $c \cdot n^{3/2}$ edges. Thus one vertex s_j must have degree at most $C\sqrt{n}$. This is the required set S_j.

4. Let $t(j)$ denote the number of divisors of the number j. Give an expression for the number $\sum_{i=1}^{n} t(j)$.

   Solution: Double-counting. Instead of counting the number of divisors of j, count how many numbers in the set $\{1, \ldots, n\}$ are divided by j. Then double-count the required sum in terms of the above quantity.

5. Given a set P of n points, and a set L of n lines in the plane, an incidence is a pair (p, l), where $p \in P$, $l \in L$, and the point p lies on the line l. Prove that given *any* set of n distinct lines L and n distinct points P, the number of incidences are at most $3n^{1.5}$.

   Solution: Exactly the same proof as done in the class for circles. Construct a bi-partite graph $G = (P \cup L, E)$, where P represents the n points, L represents the n lines, and there is an edge between $p \in V_1$ and $l \in V_2$ if the point p lies on the line l. Then note that this graph is $K_{2,2}$-free.

Bonus Problem. You are given a set P of 10 integers from the set $\{1, \ldots, 100\}$. Prove that one can always find two *disjoint* subsets of P such that the sum of the elements in the two sets are equal. For example, given the set $\{8, 15, 23, 59, 61, 70, 75, 88, 91, 97\}$, the two sets are $\{8, 15, 97\}$ and $\{59, 61\}$.

10 points.