Introduction to Combinatorics Spring, 2011

Homework 4 – Double-counting arguments

Janos Pach & Nabil Mustafa *17 March*

Questions

1. You are given a set P of n points in the plane. Prove that there exists a subset of P, say the set P' of size $m = |P'| = \Omega(\sqrt{n})$ points, with the following property. The points of P' can be ordered, denoted by the sequence $\langle p_1, \ldots, p_m \rangle$ such that the x-coordinate of the point p_i is greater than that of the point p_{i-1}, for all $i = 2 \ldots m$. And additionally one of these is true: either the y-coordinate of each point p_i is greater or equal to that of p_{i-1} for all i. Or the y-coordinate of each point p_i is less than that of p_{i-1} for all i.

   Solution: Sort the points by their x-coordinate, say they are $p_1, \ldots, p_n$. Then form the sequence where the i-th number in this sequence is the y-coordinate of p_i. Then an increasing or a decreasing subsequence is the required subset.

2. Let $\mathcal{F}$ be a family of subsets of a n-element set X. Prove that if $\mathcal{F}$ is intersecting, then $|\mathcal{F}| \leq 2^{n-1}$. Is this the best bound? If so, can you give the corresponding example.

   Solution: Note that if a set S is present, then $X - S$ is not present. This implies there can be at most 2^{n-1} sets in $\mathcal{F}$. Pick any element to be in all the sets, and the remaining elements are all possible subsets of the remaining $n-1$ elements. This gives 2^{n-1} intersecting sets.

3. Let $n \leq 2k$ and $A_1, \ldots, A_m$ be subsets of size k of $A = \{1, \ldots, n\}$, with the following property: $A_i \cup A_j \neq A$ for all i, j. Show that $m \leq (1 - \frac{k}{n})\binom{n}{k}$. (Hint: Think of the complement of each set).

   Solution: For each set A_i, call it's complement set B_i. Then B_i has size $n-k$, and they are intersecting (as $A_i \cup A_j \neq A$). Therefore, by Erdos-Ko-Rado theorem, there are at most $\binom{n-1}{n-k-1}$ such B_i's, and so $\binom{n-1}{n-k-1}$ A_i's as well. Algebraic simplification shows that this is the same as the required bound.

4. Given an integer k, let P be a set of n points such that each point has at least k points equi-distant from it. Assume no three points lie on the same line. Show that $k = O(\sqrt{n})$.

   Solution: Double-counting. Count all tuples of type (q, p_j, p_k), where $q \in P$ is equi-distant from $p_j \in P$ and $p_k \in P$. If no three points lie on the same line, then for each pair p_j, p_k, there can be at most two points $q_1, q_2 \in P$ that form the tuples (q_1, p_j, p_k) and (q_2, p_j, p_k). This gives an upper-bound of $2\binom{n}{2}$ on the number of tuples. Now get an upper-bound in terms of k. Putting them together gives the required bound.

5. Prove that the graph obtained from K_n by deleting one edge has exactly $(n-2)n^{n-3}$ spanning trees.

   Solution: Use Cayley's theorem on the number of spanning trees on n vertices, n^{n-2}. Say edge e is missing. As the number of spanning trees is n^{n-2}, and each edge appears equal number of times over these trees, the edge e must have appeared $(n^{n-2}(n-1))/\binom{n}{2}$ times. So subtract this from n^{n-2} to get the answer.

Bonus Problem. Five couples are at a party, and each person shakes hands with some of the other people, but obviously does not shake hands with their own partner. Say one of the couples is Alice and Bob. Alice then asks each of the other 9 people how many times they shook hands, and receives all distinct answers. How many people did Alice's partner Bob shake hands with?

10 points.