Questions

1. Prove that in any convex quadilateral, there exists a diagonal of length at least one-fourth the total perimeter.

   Solution: Break each diagonal into two pieces based on the intersection. Then use triangle inequality on each resulting triangle to get a lower bound on the perimeter. Then using the pigeonhole gives the bound.

2. The numbers 1 to 10 are arranged in an arbitrary order around a circle. Show that there are three consecutive numbers whose sum is at least 17.

   Solution: By contradiction. If this is not true, then each three consecutive numbers have sum at most 16. Add this up over all consecutive numbers gives the total sum of at most 160. On the other hand, in this sum, each number is counted exactly three times, to get a sum of $55 \cdot 3$, a contradiction.

3. In a contest with 3 problems, 80% of the students solve problem 1, 75% problem 2, and 70% problem 3. Prove that at least 25% of the students solve all three problems.

   Solution: Double-counting. Assuming false, the total problems solved are less than $0.75t \cdot 2 + 0.25t \cdot 3$, where t is the number of students. On the other hand, total problems solved are at least $0.8t + 0.75t + 0.7t$. This is a contradiction.

4. Given any set P of n points on the real line, and m distinct intervals on them (i.e., each interval's endpoints are from P), prove that there exists a point contained in m^2/n^2 intervals.

   Solution: Throw away all intervals containing at most $m/2n$ points. Counting shows that at most $m/2$ intervals are thrown away. The remaining $m/2$ intervals each contain more than $m/2n$ points. Now use the pigeonhole principle to argue that there is a point in $c \cdot m^2/n^2$ intervals from these $m/2$ intervals.

5. Given a set L of n lines in the plane (n is even, assume no three lines intersect at a point), show for that for any fixed point p, there exists a line l through p such that it intersects $n/2$ lines of L on both sides from p.

   Solution: Take any line l through p, and count the number of lines of L intersecting to it's right side (i.e., count lines of L intersecting l to the right of p). Say this number is t. Now rotate l by 180, and now the number of lines intersecting have become $n - t$. So somewhere on the rotation it must have been exactly $n/2$. One has to argue that during rotation, this number changes at any step by at most 1.

6. There are 30 senators in a senate. For each pair of senators, the two senators are either friends of each other or enemies of each other. Every senator has exactly six enemies. Every three senators form a committee. Find the total number of committees whose members are either all friends or all enemies of each other.

   Solution: Count the complement committees instead. Do this by iterating over each senator, and counting the number of complement committees he is in where he has one friend and one enemy. Now note that each such committie is double counted exactly twice.

Bonus Problem. Given a continuous function $f : \mathbb{S}^1 \to \mathbb{R}$ ($\mathbb{S}^1$ is the one-dimensional sphere, i.e., a circle), and any two points $p, q \in \mathbb{S}^1$, prove that one can always rotate the two points p and q around $\mathbb{S}^1$ (without changing their position relative to each other) to get the points p' and q' such that $f(p') = f(q')$.

10 points.