Introduction to Combinatorics Spring, 2011

Homework 2 – Inclusion-Exclusion Formula, Asymptotics

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Questions

1. We will prove the following equality:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \cdot 2^{n-2k} = n+1 \tag{1}$$

   - Define $\{0,1\}^n$ to be the set of all binary strings of length n. For each i, $1 \leq i \leq n-1$, define the following set:
   $$A_i = \{(x_1, \ldots, x_n) \in \{0,1\}^n : x_i = 0, x_{i+1} = 1\}$$
   A_i is the set of all binary strings of length n with 0 in the i-th position, and 1 in the $(i+1)$-th position. Prove that
   $$\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n-1} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}| = \binom{n-k}{k} \cdot 2^{n-2k}$$
   - Prove that $|\{0,1\}^n - \bigcup_{1 \leq i \leq n-1} A_i| = n+1$.
   - Prove the identity given in equation (1).

   Solution: i) Count the number of strings filling the k given places with 01's (and the rest of the places with all possible combinations). And sum this over all possible k places. ii) Counting the complement. iii) Use inclusion-exclusion.

2. How many ways are there to seat n couples in a row of $2n$ chairs such that the couples never sit next to each other?

   Solution: Inclusion-Exclusion on the complement problem.

3. How many ways are there to distribute n identical chocolates to k (non-identical!) children such that no child gets more than $m-1$ chocolates?

   Solution: Inclusion-Exclusion on the complement problem.

4. Prove the following upper bound: $n! \leq e\sqrt{n}(n/e)^n$. Use the method of integration, carefully dealing with the triangle areas.

   Solution: Exactly as done in the class, except one has to exclude the n triangle areas.

5. Prove *Bernoulli's Inequality*: for any natural number n and real $x \geq -1$: $(1+x)^n \geq 1+nx$.

   Solution: Induction on n.

6. Prove the following estimate by induction on k: $\binom{n}{k} \leq (en/k)^k$

   Solution: Induction in a very similar way to the one given in the Matousek-Nesetril book.

Bonus Problem. Let n be an even integer. Find the number of distinct strings of length n that can be obtained by concatenating copies of the strings 0, 10 and 11. For example, 0101110 is a valid string (0 10 11 10) but 1100101 is not.

10 points.

Solution: By induction. Start from the right-most bit. If that is a 0, then any string on the earlier $n-1$ bits together with this last bit 0 will form a valid codeword. So that is 2^{n-1} codewords ending with a 0. Now, if the last bit is a 1, then the second-last bit has to be a 1 also, and also has to form the 11 with the last bit (no other way for a valid codeword). This gives the recurrence $f(n) = 2^{n-1} + f(n-2)$, which solves to $(2^{n+1}+1)/3$.