Introduction to Combinatorics Spring, 2011

Homework 12 – Linear Algebra Method

Janos Pach & Nabil Mustafa *19 May*

Questions

1. Every 1-distance set in $\mathbb{R}^d$ has at most $d+1$ elements.

   Solution: Assume the same-distance is 1. Fix one point, say p_1, to be the origin. Then the remaining points, $p_2, \ldots, p_t$ all lie at distance 1 from origin. We want to show that $p_2, \ldots, p_t$ are linearly independent. Take: $\lambda_2 p_2 + \ldots + \lambda_t p_t = 0$. Multiply by p_2 to get $\lambda p_2 \cdot p_2 + \ldots + \lambda_t p_t \cdot p_2 = 0$. $p_2 \cdot p_2$ is 1, and the remaining values are $\lambda_i/2$ (from the fact that distance between p_i and p_2 is 1). Similarly, multiplying by $p_3, \ldots, p_t$, and adding up the equations, it can be seen that each $\lambda_i = 0$. Thus there can be only d points, which together with the origin forms a set of size at most $d+1$.

2. Prove Sauer's theorem by induction on n.

   Solution: See the detailed solution at: `http://www.cs.princeton.edu/courses/archive/spr08/cos511/scribe_notes/0220.pdf`

3. Prove the following analog of Sauer's Lemma for uniform families. Let n, l, k be natural numbers, $n \geq l \geq k$ and let $\mathcal{F}$ be an l-uniform family of subsets of an n-element set. If $|\mathcal{F}| > \binom{n}{k-1}$ then $\mathcal{F}$ is (n,k)-dense (i.e., there exists a subset Y of size k such that every subset of Z can be gotten from intersecting Y with the sets in $\mathcal{F}$). (Hint: follow the proof given in class for this new setting, where the columns are labelled with $(k-1)$-element subsets only, and argue that the minimal set Y, for which $g(Y) \neq 0$, must still have at least k elements.)

   Solution: Hint gives the solution sketch.

4. Construct a two-distance set in $\mathbb{R}^n$ with $\binom{n}{2}$ points. Recall that a set of points in $\mathbb{R}^n$ is two-distance iff the distance between every pair of points is one of two values.

   Solution: A point is described by n coordinates. For each pair of coordinates, make a point by setting those two coordinates to 1, and the other $n-2$ coordinates to 0. These are the $\binom{n}{2}$ points, and it can be verified that this is a two-distance set.

Bonus Problem. An $n \times n$ table filled with integers has the property that no two columns are identical. Then prove that there exists a row which can be removed so that the property is still true.

10 points.