Introduction to Combinatorics Spring, 2011

Homework 11 – Linear Algebra Method

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Questions

1. Prove the "reverse oddtown theorem": given a set X of n elements, and m sets $S_1, \ldots, S_m$ over X where each $|S_i|$ is even, and $|S_i \cap S_j|$ is odd, prove that $m \leq n+1$. Can you improve this to $m \leq n$?

   Solution: Add a dummy element to each set, and apply Oddtown theorem. Therefore, $m \leq n+1$. It can also be shown that $m \leq n$. If n is odd, take the complement sets, and apply Oddtown theorem. Otherwise, n is even. Now note that the set system made up of $\{S_i, i = 1 \ldots m\}$, and the complement set system made up of sets $\{\overline{S_i}, i = 1 \ldots m\}$ satisfying same condition. And for contradiction, assume $m = n+1$. Then by making the equation for linear dependency on the characteristic vectors of both set S_i, and the complement set $\overline{S_i}$ (following the proof done in class), it can be deduced that $v_i + \overline{v_i} + \ldots + v_{n+1} + \overline{v_{n+1}} = 0$, a contradiction as $n+1$ is odd, and so the above sum is 1, and not 0.

2. Prove the "bipartite oddtown theorem": given a set X of n elements, and sets $R_1, \ldots, R_m$ and $B_1, \ldots, B_m$, where $|R_i \cap B_i|$ is odd for every i, and $|R_i \cap B_j|$ is even for every $i \neq j$, prove that $m \leq n$.

   Solution: Say characteristic vector of R_i is r_i, and B_i is b_i. Then make the linear-dependency equation for r_i. To see $\lambda_i = 0$, multiply equation by b_i.

3. A block-design consisting of n total elements X and m sets over X where each set has k elements and every t-sized subset of X is contained in exactly λ sets is denoted by t-(n, k, λ). Then Fisher's inequality states that $m \geq n$.

   Prove that for a block-design with $t = 2$, $\lambda \cdot \frac{n-1}{k-1}$ has to be an integer.

   Solution: Say the first element lies in c sets. Then for each of those sets, it makes a pair with the other $k-1$ elements, for a total of $c(k-1)$ pairs. On the other hand, it makes a pair with $n-1$ elements, and each of those pairs appears in λ sets. So then $c(k-1) = \lambda(n-1)$, and we're done.

4. Can there be a block-design of type 2-$(16, 6, 1)$? What about 2-$(21, 6, 1)$? 2-$(25, 10, 3)$?

   Solution: None are possible. First note that by double-counting, we have $m = \lambda \frac{n(n-1)}{k(k-1)}$. Now apply Fisher's inequality to get a contradiction on m.

5. For $\lambda = 1$, prove Fisher's inequality directly.

   Solution: Take any set A with k elements. As each element lies in $\frac{n-1}{k-1}$ sets, then for each element $i \in A$, there are $(n-1)/(k-1) - 1$ other sets containing i. Over all $i \in A$, these sets are distinct, so the total number of sets are at least $(\frac{n-1}{k-1} - 1)k + 1$. It can be verified that this is at least n.

Bonus Problem. Suppose we have a necklace of n beads. Each bead is labelled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \ldots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k-1 \quad \forall\, k = 1, 2, \ldots, n$$

10 points.