Introduction to Combinatorics Spring, 2011

Homework 10 – Probabilistic Methods

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Questions

1. Let X be a random variable, and $c > 0$ any constant. Prove that $Var[cX] = c^2 Var[X]$.

   Solution: $V[cX] = E[(cX - E[cX])^2] = E[(cX - cE[X])^2] = c^2 E[(X - E[X])^2] = c^2 V[X]$

2. Prove that the Crossing Lemma is optimal. In other words, given any integers $n > 0$ and $m \geq 5n$, show that there exists a graph G with n vertices and m edges such that the crossing number of G is at most $c \cdot m^3/n^2$, where $c > 0$ is a constant.

   Solution: Make a graph on n vertices by dividing these n vertices into n^2/m equal-sized sets, each containing m/n vertices. In each set, make a clique on these m/n vertices. This is the required graph. Since each set is a complete graph, it has crossing number at least $(m/n)^4$, and there are n^2/m such cliques.

3. In class we saw a probabilistic proof of the Crossing Lemma. Using that for intuition, construct a purely combinatorial double-counting proof of the Crossing Lemma.

   Solution: Consider the subgraphs induced by each subset of size $t = cn^2/m$, where c is a suitable constant. Say the i-th subset has vertices V_i and induced edges E_i to make the induced graph G_i. Then the weak crossing Lemma implies $cr(G_i) \geq |E_i| - 3|V_i|$. Sum this up over all $\binom{n}{t}$ graphs (using double-counting to count the various sums) to get the crossing lemma.

4. In a way similar to the one done in class, prove that the number of incidences between n distinct unit circles and n distinct points in the plane is at most $c \cdot n^{4/3}$, where $c > 0$ is a constant.

   Solution: This is exactly same way as Point-Line incidences is gotten using crossing lemma. Except that now, between two points there could be two edges. But then simply throw away any one of those edges. The remaining graph has still at least $m/2$ edges, if m was the number of edges in the original graph. The rest is same as done in class.

Bonus Problem. A deck of 50 cards contains two cards labeled i for each $i = 1, 2, \ldots, 25$. There are 25 people seated at a table, each holding two of the cards in this deck. Each minute every person passes the lower-numbered card of the two they are holding to the right. Prove that eventually someone has two cards of the same number.

10 points.