Introduction to Combinatorics Spring, 2011

Homework 1 – Basic Counting

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Questions

1. Suppose that n teams play a tournament in which every team plays every other team exactly once, and there are no tie games. Prove by induction that, no matter what the individual game outcomes are, one can always number the teams $t_1, t_2, \ldots, t_n$, so that t_1 beat t_2, t_2 beat t_3, and so on, through t_{n-1} beat t_n.

   Solution: Induction. Assume teams $t_1, \ldots, t_{n-1}$ are in the required order. Then we only have to add t_n to this order. Then one has to show that there exists an i such that t_i beats t_n and t_n beats t_{i+1}. Then we insert t_n after t_i and before t_{i+1}.

2. How many ordered pairs (A, B), where A, B are subsets of $\{1, 2, \ldots, n\}$, are there such that $|A \cap B| = 1$?

   Solution: n ways to pick the element common to A and B. Then for each of the $n-1$ remaining elements, there are three choices: either in A, or in B, or in neither. So total pairs: $n3^{n-1}$.

3. Consider a set A of n elements. Count the number of sequences of the form $(A_1, A_2, \ldots, A_k)$ where $A_i \subseteq A_{i+1}$ and $A_k \subseteq A$.

   Solution: For each element $x \in A$, assign it the number corresponding to the highest i such that $x \in A_i$ and $x \notin A_{i+1}$. Each such assignment to A gives a corresponding sequence of sets, and vice versa. So the answer is: k^n.

4. Prove the following with a combinatorial argument:

$$\begin{aligned}
\binom{n+m}{2} &= \binom{n}{2} + \binom{m}{2} + nm \\
\sum_{k=0}^{n} \binom{x}{k} \cdot \binom{y}{n-k} &= \binom{x+y}{n} \\
\binom{\binom{n}{2}}{2} &= 3 \cdot \binom{n}{4} + 3 \cdot \binom{n}{3} \\
\sum_{k} \binom{n}{2k} \cdot \binom{2k}{k} \cdot 2^{n-2k} &= \binom{2n}{n}
\end{aligned}$$

   Solution: i) Count numbers of pairs from $A \cup B$, where $|A| = n$ and $|B| = m$. ii) Count how many ways to pick n elements out of a set of size $x + y$. iii) Count pairs of pairs over n base elements. iv) One way to pick n elements out of $2n$ is to divide $2n$ elements into n pairs, and then see what happens to these pairs if we pick n total elements.

5. Prove that the coefficient of x^k in $(1 + x + x^2 + x^3)^n$ is:

$$\sum_{i=0}^{k} \binom{n}{i} \cdot \binom{n}{k-2i}$$

   Solution: $(1 + x + x^2 + x^3) = (1 + x)(1 + x^2)$. Now use the same method as in the proof of Binomial theorem.

6. Prove the following identity in three different ways, with a combinatorial argument, an inductive argument and using the binomial theorem:

$$\sum_{i=0}^{n} \binom{n}{i} \cdot 2^i = 3^n$$

Solution: One way is to use Binomial theorem. Another way is to use induction. A third way is a counting argument.

Bonus Problem. Two players A and B have to divide a apple-pie among themselves. B divides the apple-pie into $2n$ (possibly unequal) slices, where n is any positive integer. Then A picks any slice from the $2n$ slices for himself. Then B picks one of the two slices which were next to A's slice. Then A picks a slice, then B and so on. The only restriction is that each time A or B pick a slice, the remaining apple-pie must consist of consecutive slices (i.e., all the slices picked so far must be consecutive). Prove that no matter how B divides the apple-pie, A can always get at least half the apple-pie for himself.

10 points.

Solution: Number the slices from 1 to $2n$. Then either the even-numbered or odd-numbered slices will have at least half the total apple-pie. Show that A can always get all the even-numbered or all the odd-numbered slices, whichever is greater.