On a planar matching in line segments endpoints visibility graph

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Abstract

We show that the *segment endpoint visibility graph* of any finite set of $2n$ line segments contains a planar matching that does not use any segment edge, if the line segments in our set are *convexly independent*.

1 Introduction

We start with the description of the terminology. By a *matching* in a graph $G(V, E)$ we understand a subset of edges $E' \subseteq E$ such that no pair in E' share a vertex and such that every vertex in V is incident to one edge in E'. If $G(V, E)$ is a plane graph (abstract graph with its embedding in plane), by *planar matching* we understand a matching in G none of which pair of edges cross.

The *segment endpoint visibility graph* $\text{Vis}(S) = G(V, E)$ for a set of n disjoint closed line segments is a plane graph defined as follows. The set of vertices V consists of the endpoints of the line segments in S and two vertices v_1 and v_2 in V are joined by edge belonging to E iff they are the endpoints of the same line segment in S or if the line segment connecting them does not cross any other line segment in S. In the former case we call such an edge a *segment edge* and in the latter a *visibility edge*. Where convenient we will refer to a segment edge by the identifier of the corresponding line segment. Let us denote by $E_s(S)$ and $E_v(S)$ the subsets of E containing segment and visibility edges, respectively. We denote by $\text{Vis}'(S)$ the subgraph of $\text{Vis}(S)$ on the same vertex set and containing all visibility edges.

We say that the set S of line segments in plane is *convexly independent* if every line segment in S has at least one point on the boundary of the convex hull of the line segments in S.

We will prove the following theorem.

Theorem 1. *Let S be a convexly independent set of $2n$ disjoint line segments in the plane in the general position. Then there exists a planar matching in $\text{Vis}'(S)$.*

2 Basic observations

Given a finite set S of disjoint line segments in the plane. One can easily see that a matching M in $\text{Vis}'(S)$ partition the set of line segments S into k parts such that each part gives rises to a cycle in $\text{Vis}(M)$ whose edges alternates between visibility and segment edges as follows.

Observation 2. $S = \cup_{i=0}^{k-1} S_i$, $M = \cup_{i=0}^{k-1} E_i$ *such that* $E_s(S_i) \cup M_i$ *is a cycle.*

Thus, we can reduce our question of finding a matching in $\mathrm{Vis}'(S)$ to a question of finding a disjoint alternating cycles that cover all vertices in $\mathrm{Vis}'(S)$.

The other simple observation is that we can separate the set S of $2n$ line segments into two non-empty parts S_1 and S_2 each of which contains an even number of segments, if there is $s \in S$ such that the line l containing s intersects any other $s' \in S$, and both open half-planes defined by l contain at least one line segment from S. We define the set S_1 as a set containing all line segments in an open half space L' defined by l, if the number of these line segments is even, and as a set containing s, and all line segments in L' otherwise. Then we obtain S_2 as a complement of S_1 in S.

Observation 3. *The union of matchings M_1 and M_2 in $\mathrm{Vis}'(S_1)$ and $\mathrm{Vis}'(S_2)$, respectively, is a matching $M = M_1 \cup M_2$ in $\mathrm{Vis}'(S)$. Moreover, if M_1 and M_2 are planar then also M is planar.*

3 Algorithm

In this section we describe an algorithm that for a given set S of $2n$ disjoint line segments in the plane in the general position, that are convexly independent outputs a planar matching in $\mathrm{Vis}'(S)$.

In the light of Observation 3 from now on we assume that there is no $s \in S$ not fully contained on the boundary of the convex hull of S such that the line l containing s does not share a point with any other $s' \in S$, and both open half-planes defined by l contain at least one line segment from S. Otherwise we can divide our problem into smaller subproblems such that each subproblem satisfies this condition, solve these subproblems separately and in accordance with Observation 3 output the solution.

Let us denote the line segments in $S = \{s_0, \dots s_{2n-1}\}$ such that their indices correspond to the order of the appearance of their endpoints on the boundary of their convex hull $C = \mathrm{conv}(S) = \mathrm{conv}(\{s_0, \dots s_{2n-1}\})$. Note, that by the above assumption we do not have any diagonal line segment with respect to C among the elements of S. We denote this cyclic order of the line segments in S by $O(S)$. We denote by $(s_i, 0)$, $i \in \mathbb{N}$, the endpoint of s_i on the boundary of C, and by $(s_i, 1)$ the other endpoint of s_i such that if both endpoints of s_i belongs to the boundary of C traversing the boundary of C in the positive direction from $(s_i, 0)$ to $(s_i, 1)$ takes place in s_i. If $r = (s_i, p)$, by r_s we will understand s_i.

For an endpoint (s_i, p), $i \in \mathbb{N}$, $p \in \{0, 1\}$ and direction $d \in \{+, -\}$ we define $\mathrm{next}((s_i, p), d, k)$, $k \in \mathbb{N}$, as the k-th neighboring endpoint in the circular order of the neighbours of (s_i, p) in $\mathrm{Vis}(S)$ defined by its embedding starting at the endpoint $(s_i, (p+1) \mod 2)$ in direction d ($+$ for positive and $-$ for negative direction). If $k = 0$ the function next returns s_i. Analogously we define $\mathrm{next}((s_i), d)$ and $\mathrm{prev}((s_i), d)$ as the next and previous line segment in the direction d in the cyclic order $O(S)$.

In what follows we describe the procedure GetAlternateCycle(S), which on the input S returns a set of disjoint alternating cycles $\mathcal{C}$ (i.e. the edges on it alternate between segment and visibility edges) in the graph $\mathrm{Vis}(S)$. Let $S_\mathcal{C}$ denote the subset of S containing the line segments on cycles in $\mathcal{C}$. Let us denote two consecutive (with respect to $O(S)$) segments of $S_\mathcal{C}$ by s_i and s_j, $0 \le i, j < 2n$. We will construct $\mathcal{C}$ in a way that we can partition any set $\{s_{i+1 \mod 2n}, \dots s_{j-1 \mod 2n}\}$ (if non-empty) into the parts $S_1, \dots S_m$, $1 \le m$, each of which consists of an even number of consecutive line segments in our cyclic order $O(S)$. Moreover, the convex hull of each part S_i, $1 \le i \le m$, and therefore also $\mathrm{Vis}(S_i)$, is disjoint from the plane graph $\mathrm{Vis}(S \setminus S_i)$. By proving that this procedure is correct we finish almost all work needed in order to prove Theorem 1. Indeed, it allows us to

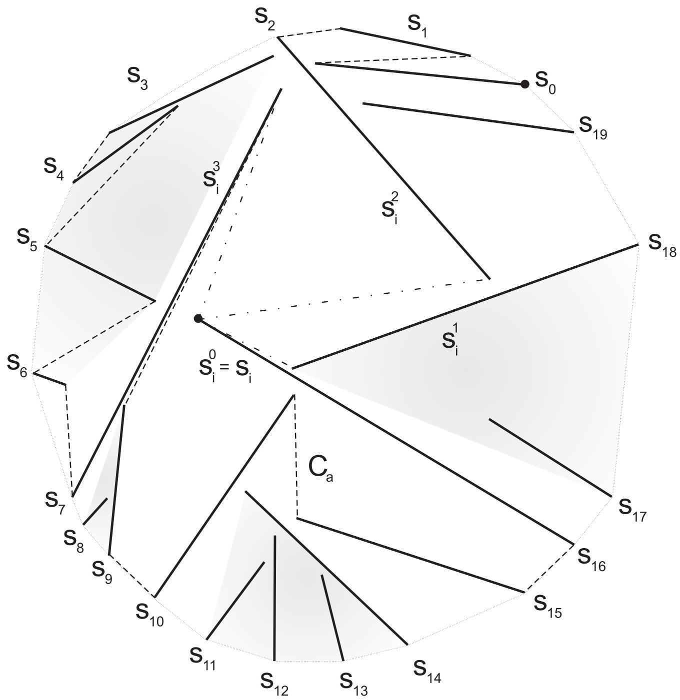

Figure 1: Illustration of a step in GetAlternateCycle(S), $i = 16$, $k = 3$, $j = 7$, $d = +$

divide our problem into smaller independent subproblems. Thus, the claim of Theorem 1 then follows by induction on the number of line segments in S.

We initialize an output as an alternating path C_a, and direction d as follows $C_a := (s_0, 0), (s_0, 1)$, and $d = +$.

Let (s_i, p), for some $0 \leq i < 2n$, be the last vertex on (so far) the path C_a. We choose as the next next vertex on C_a $(s_j, p') = \text{next}((s_i, p), d, k)$, $k \in \mathbb{N}$, where $k = \min K = \{k' | \ (s_j, p) = \text{next}((s_i, p), d, k'),\ s_j - s_i \equiv 1 \pmod 2\}$. Let s_i^j denote $\text{next}((s_i, p), d, j)_s$ (see Figure 1 for an illustration). Observe that because of the minimality of k each set of the line segments of the form $\{\text{next}(s_i^{j-1}, d), \dots s_i^j\}$, $1 \leq j \leq k-1$, and $\{\text{next}(s_i^{k-1}, d), \dots \text{prev}(s_i^k, d)\}$ contains an even number of elements. Moreover, it is easy to see (nevertheless we will show it later) that a convex hull of any of these sets is disjoint from the rest of our visibility graph. At each step of the procedure we consider the following cases.

a) If (s_j, p') is already on C_a and its first appearance is followed by $(s_j, p'+1 \pmod 2)$ (see Figure 2 for an illustration) we output the part of C_a between the first and the second appearance of (s_j, p'), i.e. $(s_j, p'), (s_j, p'+1 \pmod 2) \dots, (s_j, p')$.

b) If (s_j, p') is already on C_a, but its first appearance is not followed by $(s_j, p'+1 \pmod 2)$ (see Figure 3 for an illustration) we reverse the part of C_a between (s_i, p) and (s_j, p') remove the last endpoint (s_j, p') and set it as C_a , i.e. $C_a := (s_i, p), \dots r = \text{next}((s_j, p'), d, k')$, for appropriate k', we set $d := -d$ and we proceed as follows with r being the last vertex on our path (see Figure 4 for an illustration). Later we show that in this case $p' = 1$.

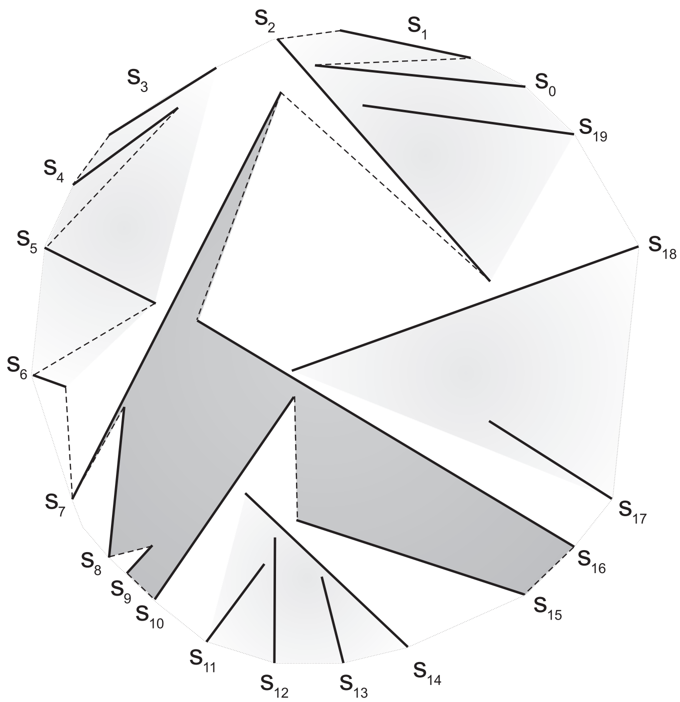

Figure 2: Illustration of the case a) in GetAlternateCycle(S), $i = 16$, $k = 3$, $j = 7$, $d = +$

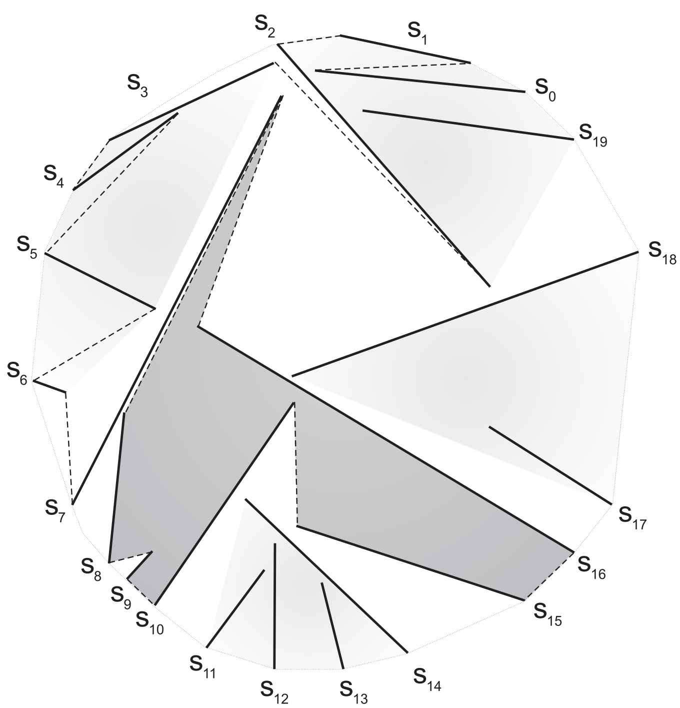

Figure 3: Illustration of the case b) in GetAlternateCycle(S), $i = 16$, $k = 3$, $j = 7$, $d = +$

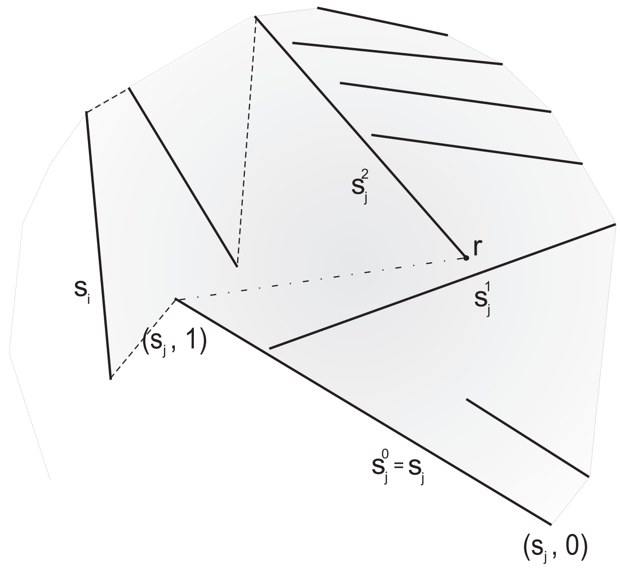

Figure 4: Illustration for the case b) in GetAlternateCycle(S)

1. If one of two cases a) or c) happens we do exactly the same thing.
2. Let s_i, k be as above. If b) happens. Let $k'' = \min\{k' \in K|\ \mathrm{next}(s_i, d, k') \in C_a\}$. We do everything as in the step b), except that we also run this subroutine (steps 1. and 2.) on $\{\mathrm{next}(s_i^{k''-1}, d), \ldots s_i{}^{k''}\}$, if $k'' < k$, and on $S' = \{\mathrm{next}(s_i^{k-1}, d), \ldots \mathrm{prev}(s_i{}^k, d)\}$, otherwise, such that $d := d$ and C_a is initialized as C_a at this steps with just created loop removed. Thus, the last vertex on C_a is the first vertex on C_a before the first appearance of $\mathrm{next}(s_i, d, k'')$ not belonging to the same segment. Notice, that at this point we divided C_a into two parts (see Figure 5 for an illustration).

Notice that the convex hull of the union of the line segments s_i and s_j does not have to be necessarily disjoint from the rest of Vis(S), i.e. from Vis($S \setminus \{s_i, s_j\}$). Thus, one should prove that we are always able to prolong C_a to $(s_j, p' + 1 \mod 2)$ followed by (s_j, p'), and enclose it by visiting (s_i, p). Indeed, otherwise we might have to run our algorithm on the subset of S that is not disjoint from the rest of Vis(S). The purpose of the case 2. from the above is to prolong C_a according to the discussion.

c) Otherwise we set $C_a := C_a, (s_j, p'), (s_j, p' + 1)$ and proceed to the next step.

Let us call the above described procedure GetAlternateCycle(S). The main algorithm (Algorithm 1) works as follows.

1. We obtain the partition of S according to Observation 3, $S_1, \ldots S_k$
2. For each S_i, $1 \le i \le k$, we run GetAlternateCycle(S_i) thereby obtaining a set of alternating cycles $\mathcal{C}$. We run Algorithm 1 on each non-empty set of segments of the forms $\{\mathrm{next}(s_i, d), \ldots s_i^1 = \mathrm{next}((s_i, p), d, 1)_s\}$, $\ldots \{\mathrm{next}(\mathrm{s_i^1}, \mathrm{d}), \ldots (s_i^2, d)\}$, $\ldots \{\mathrm{next}(\mathrm{s_i^{k-1}}, \mathrm{d}), \ldots \mathrm{prev}(s_i^k, d)\}$, such that $\mathrm{next}((s_i, p), d, k)$ is the considered vertex from above during any step of our algorithm.

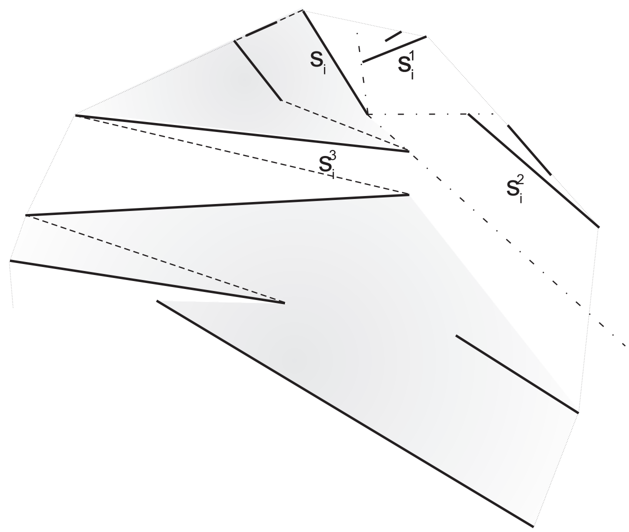

Figure 5: The division of C_a into two parts

3. We output the union of the visibility edges of the cycles in $\mathcal{C}$ and the edges that were returned by all recursive calls of Algorithm 1.

4 Proof of Theorem 1

In this section S is again a set of disjoint line segments $S = \{s_0, \dots s_{2n-1}\}$ in the plane and in the general position.

Let $(s_i, 1)$, k be as in the GetAlternateCycle(S) procedure. We denote by p_l, $1 \leq l \leq k$ a point we get as the intersection of the ray from $(s_i, 1)$ through $\mathrm{next}((s_i, 1), d, l)$ with the first line segment from S it intersects, or with the convex hull of S in the case that it does not intersect any line segment. Let H_l denote the convex hull of $(s_i^l, 1)$, $(s_i^l, 0)$, and p_l in the case that p_l is not contained in the interior of any line segment of S, and of $(s_i^l, 1)$, $(s_i^l, 0)$, p_l and $(s', 0)$, s.t. $p_l \in s'$, otherwise. In fact s' has to be s_i^{l-1}.

In order to proof our theorem first we will need the following observation.

Let S' be one of the sets of line segments $\{\mathrm{next}(s_i^{j-1}, d), \dots s_i^j\}$, $1 \leq j \leq k-1$, or $\{\mathrm{next}(s_i^{k-1}, d), \dots \mathrm{prev}(s_i^k, d)\}$, where $\mathrm{next}((s_i, p), d, k)$ is considered vertex from above during any step of GetAlternateCycle(S) procedure. We denote by S'' the union of these sets.

Observation 4. *The convex hull of each set S' is disjoint from* $\mathrm{Vis}(S \setminus S'')$.

Proof. Notice that p has to equal to one, if there is a non-empty set S' of the desired form.

One can easily see that all line segments $\{\mathrm{next}(s_i^{j-1}, d), \dots s_i^j\}$, $1 \leq j \leq k-1$, for $l < k$, and $\{\mathrm{next}(s_i^{k-1}, d), \dots \mathrm{prev}(s_i^k, d)\}$, for $l = k$, are contained in H_l. Indeed, if there were a line segment not fully contained in any of these convex hulls we would obtain at least one new neighbor of $(s_i, 1)$ in $\mathrm{Vis}(S)$. One can convince himself by taking the convex hull H of the parts of the line segments between ${s_i}^{l-1}$ and ${s_i}^l$ not contained in H_l for some l. H has to belong to the wedge with apex (s_i, p)

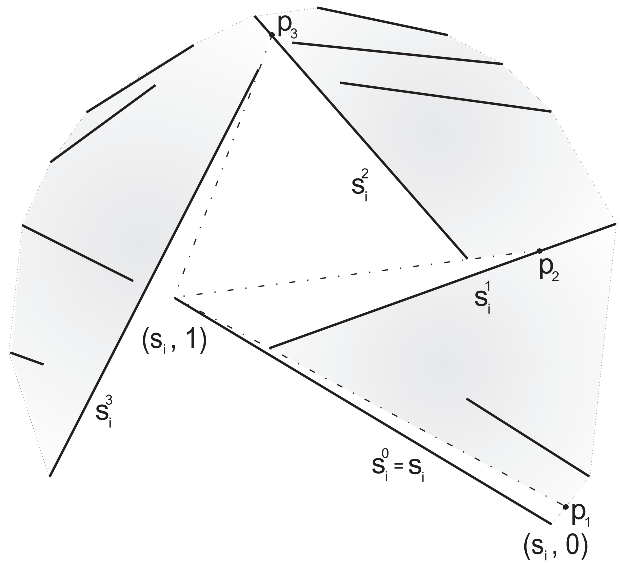

Figure 6: Illustration of Observation 4

and boundary rays through $u = \mathrm{next}(s_i^{l-1}, d)$ and $v = \mathrm{next}(s_i^l, d)$. Otherwise one of the visibility edges between (s_i, p) and u, and (s_i, p) and v would be blocked. Obviously, the point w on the boundary of H minimizing the angle $\measuredangle u(s_i, 1)w$ has to be an endpoint of a line segment S whose visibility from $(s_i, 1)$ is not blocked by anything yielding contradiction.

Now, it is easy to see that there is no edge of $\mathrm{Vis}(S \setminus S'')$ having non-empty intersection with corresponding H_l as in this case at least one of its endpoint would belong to H_l. $\square$

By the proof of Observation 4 we have the following.

Observation 5. *The convex hull of each set S' is contained in a corresponding H_l, for some $1 \leq l < k$. Hence, their visibility endpoints graphs are pairwise disjoint.*

By the above observations we have justified the cases a) and c) of GetAlternateCycle(S) procedure. So, all what is left to prove is that if we flip the direction in the case b), we will eventually arrive at $(s_j, p' + 1 \mod 2)$ followed by (s_j, p'). We present several simple observations, which turn out to be enough to prove this.

The proof of the following observation is rather obvious.

Observation 6. *At the begining of any step during* GetAlternateCycle(S) *procedure the order of the segment edges on C_a corresponds to the order $O(S)$, i.e. for all pairs of two consecutive segment edges on C_a, s_i and s_j, $0 \leq i, j < 2n$, such that s_j follows s_i if we traverse C_a in the positive direction, we have that $s_{i+1 \mod 2n}, \ldots s_{j-1 \mod 2n} \notin C_a$.*

Proposition 7. *If the case b) during the execution of* GetAlternateCycle(S) *happens, both (s_j, p') and (s_i, p) are not on the boundary of the convex hull of S.*

Proof. Clearly, (s_i, p) is cannot be on the boundary of the convex hull of S. Now, we proceed by contradiction (see Figure 7 for an illustration). Let (s_i, p) and $(s_j, p' = 0)$ be as at the begining

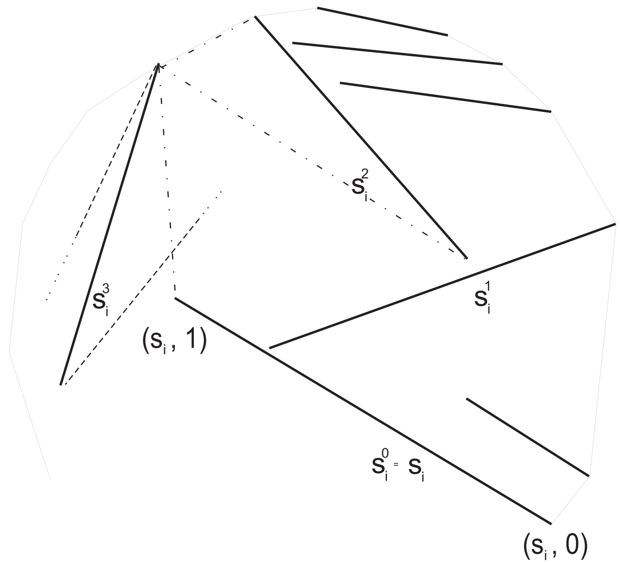

Figure 7: Illustration for the proof of Proposition 7

of a step during the execution of GetAlternateCycle(S) such that the case b) happens. Obviously $(s_j, 0)$ cannot be the first vertex on C_a as that would mean that the case a) happens. Hence, there is by Observation 6 a line segment $s' \in S$ between s_i and s_j with respect to the direction d, which is the segment s' on C_a followed by s_j. The endpoint on s' preceeding $(s_j, 1)$ is $(s', 1)$ as otherwise the endpoint, that follows it, would be $(s_j, 0)$. Clearly, $s' \in S'$, for some S', where S' is the same as in Observation 4.

The line segment s' cannot be s_j^l, for any $1 \leq l \leq k$, since that would mean that $(s_j, 1)$ cannot be the endpoint following $(s', 1)$. Indeed, it is clearly not possible when $s' = \text{prev}(s_j, d)$ (in this case $(s_j, 0)$ would be the following vertex on C_a), and otherwise by Observation 5 $(s', 1)$ could not belong to the region bounded by s_i, the line segment $(s_i, 1)(s_j, 0)$ and the part of the boundary of the convex hull of S between $(s_i, 0)$ and $(s_j, 0)$ with respect to d.

On the other hand if $s' \neq s_j^l$, for all $1 \leq l \leq k$, by Observation 5 there is no visibility edge between an endpoint of s' and $(s_j, 0)$. Thus, we have arrived at contradiction. □

Proof. (Theorem 1)

In order to prove that after the case b) in GetAlternateCycle(S) procedure happens our subroutine (cases 1. and 2.) prolongs C_a in a desired way, it is enough to show following. If during the execution of the subroutine we encounter the line segment s_j^l, for some $1 \leq l \leq k$, as at the begining of the step when we call the subroutine, the case that corresponds to c) in GetAlternateCycle(S) happens, i.e. s_j^l will be just the next segment in the ordering $O(S)$ with respect to d.

We fix J such that s_J is equal to s_j at the begining of the step when we call the subroutine (cases 1. and 2.), and we fix d as at the begining of the step when we call the subroutine as well. So, let the position of the last segment edge of C_a during the execution of the subroutine be between $r = s_J^l$ and $r' = s_J^{l-1}$ with respect to d.

If the case corresponding to c) does not happen and $s_i^{l'}$, for some $1 \leq l' \leq k$, is r, then r' is contained either in a subset S' or in S'', where S' is one of the sets of line segments

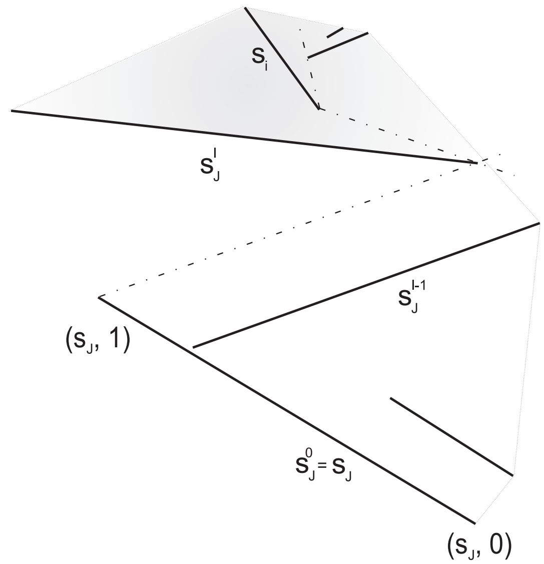

Figure 8: Illustration of impossibility of the situation when r' is contained in S''

$\{\text{next}(s_i^{j-1}, d), \ldots s_i^j\}$, $1 \leq j \leq k-1$, $j \neq l'$, and $\{\text{next}(s_i^{k-1}, d), \ldots \text{prev}(s_i^k, d)\}$ if $l' \neq k$, where $\text{next}((s_i, p), d, k)$ is the considered vertex from above during any step of GetAlternateCycle(S) procedure, and S'' is either $\{\text{next}(s_i^{k-1}, d), \ldots \text{prev}(s_i^k, d)\}$, if $l' = k$, or $\{\text{next}(s_i^{l'-1}, d), \ldots (s_i^{l'}, d)\}$, $l' \neq k$, otherwise. We show that r' must be contained contained in S''.

Indeed, as otherwise by Observation 5 r' cannot be distinct from $s_i^{l''}$ (because its visibility from $(s_J, 1)$ would be blocked), for all $1 \leq l'' \leq k$, $l'' \neq l'$, and by the similar argument as in the proof of Proposition 7, r' cannot be one of $s_i^{l''}$, for $1 \leq l'' \leq k$, $l'' \neq l'$.

So, r' is contained in S''. However, this is also impossible because that would mean that the line containing r does not intersect any other line segment in S (see Figure 8 for an illustration).

Thus, we can encouter s_J^l only in the case c), since we can encouter it only as a following segment in the ordering $O(S)$ with position before s_J with respect to d. And therefore as a segment that have not been visited yet. Finally, as we are unable to make a cycle using one of s_J^l eventually we have to arrive at $(s_J, 0)$ and by Proposition 7 we can enclose the cycle by adding $(s_J, 0)$ and $(s_i, 0)$. $\square$

References