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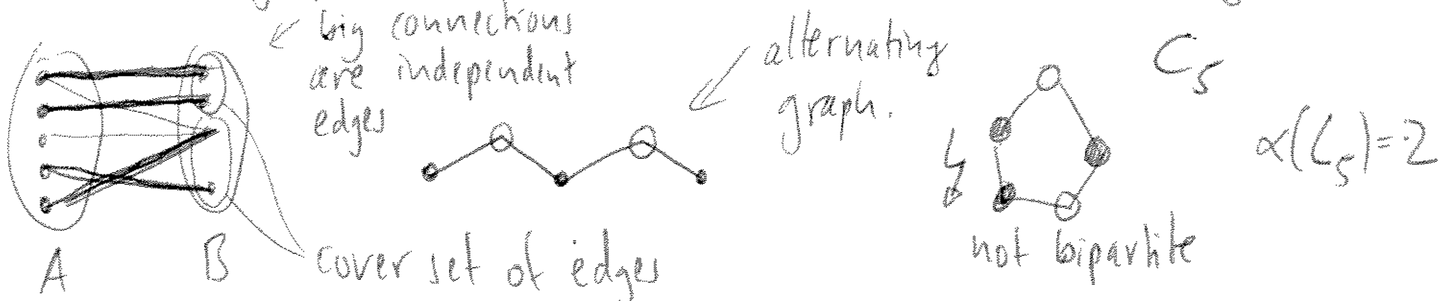
Alternative book: Bondy-Murty: Introduction To Graph Theory (old book!) ≈ 1970

Oral exam: One or two problems of problem sheets need to be solved there.

Matchings in Bipartite Graphs

Def.: A graph is an ordered pair of vertices $V(G)$ and edges $E(V)$.
E.g. $V(G) = \{1,2,3,4\}$ and $E(G) = \{12, 23, 34\}$.

Def.: A bipartite graph is such that $V(G) = A \cup B$, e.g.



Proposition: ~~Proposition~~: A graph is bipartite iff $V(G) = A \cup B$ and A and B are independent sets (i.e. no edges within A and no edges within B).

Def.: $\alpha(G) :=$ max # vertices in an independent set of G

To determine $\alpha(G)$ is computationally intractible, an NP-complete problem.

Def.: Matching = set of independent edges (i.e. no two edges share a vertex.

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Thm.: (König)

Let G be a bipartite graph, i.e. $V(G) = A \cup B$, A, B independent vertex sets.

(The maximal size of a matching $M \subseteq E(G)$)

$=$ min # vertices that cover all edges of G.

Where "cover" means that every edge contains at least one of G's vertices.

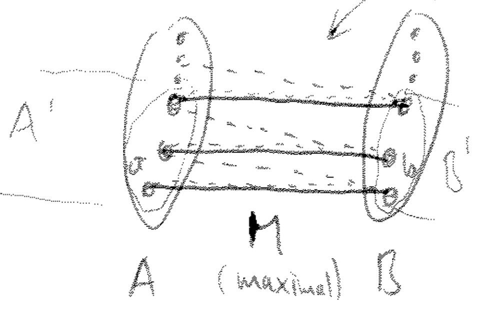
(Methatheorem: (Empiracal fact, not a theorem!)

Whenever we have a max $=$ min theorem then computing this parameter is computationally easy (i.e. possible to do in polynomial time n).)

Proof: max $|M| \leq$ min # vertices that cover all edges of G is trivial to see. So max $|M| \geq$ min # vertices that cover all edges of G.

Def.: Let M be a fixing. An alternating path is a path starting at an unmatched vertex of A and having every other edge in M.

pointed lines are an alternating path.



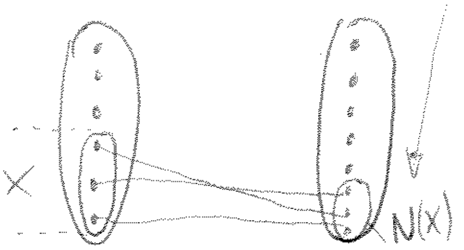
Let $ab \in M$. Pick b if there is an alternating path ending at b, pick a otherwise. (*)

this guarantees that no edges go to the points not in the Matching M.

We claim that the vertices we picked cover all edges of G. The solid (matching) edges are covered clearly but also the pointed ones? The edges from $A \setminus A'$ to B' are fine. (covered). There are no edges going from $A \setminus A'$ to $B \setminus B'$ and also none that go from A' to $B \setminus B'$. But also the edges from A' to B' are covered which is clear when applying rule (*).

neighbours of set X. $\square$

Thm.: (Hall)



$|A| \leq |B|$

There is perfect matching of A iff ($\Leftrightarrow$ $|A| \leq |B|$, and) $\forall X \subseteq A$, $|N_G(X)| \geq |X|$.

$N_G(X) := \{ b \in B \mid$ b is connected to at least one element in $X \}$

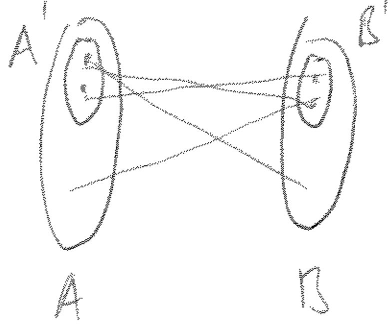
Proof: König's theorem implies Hall's theorem.

"$\Leftarrow$" Suppose $\forall X \subseteq A$ $|N(X)| \geq |X|$. It is to show that this allows a perfect matching of A. I.e. that the maximum size of a mathing $= |A|$.
By König's theorem, this max size of a matching is the minimum size of a vertex cover. By contradiction:

Assume that $|A' \cup B'| < |A|$ is a vertex cover, where $A', B' \subseteq A, B$ E.g.

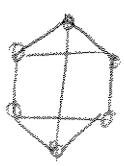
Contradicting König's thm.

every edge ends in A' or B'.



But then $A \setminus A'$ will violate $(*)$, namely $\forall X \subseteq A$ $|N(X)| \geq |X|$. Because $|N(A \setminus A')| \leq |B'| < |A \setminus A'|$ ↯, contradiction. $\times$ (One could also prove that by induction).

Corollary I: Def.: A graph G is said to be k-regular if $\forall v \in V(G)$, $\deg(v) = k$ ($\deg(v) =$ # edges incident to v = # neighbors of v)

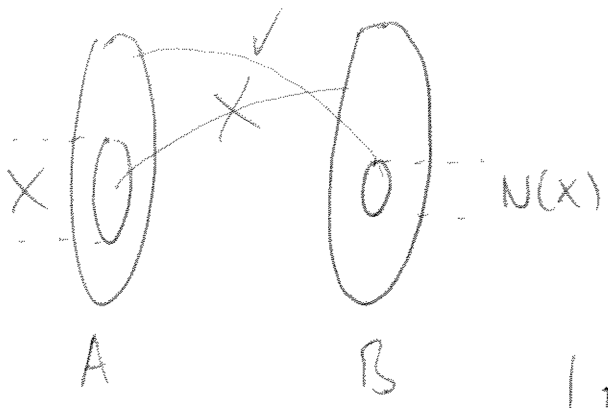
E.g.  is 3-regular.

Let G be a k-regular bipartite graph and let $V(G) = A \cup B$. Then $|A| = |B|$ and there is a perfect matching.

Proof: $\sum_{v \in A} d(v) = \sum_{v \in B} d(v) = |E(G)|$ so, by k-regularity,

$k|A| = k|B| \Rightarrow |A| = |B|$.

It remains to show that $\forall X \subseteq A \quad |N(X)| \geq |X|$.



$$\Rightarrow \underbrace{\sum_{v \in X} d(v)}_{|E(X, N(X))|} \leq \sum_{v \in N(X)} d(v)$$

So $|X| \cdot k \leq |N(X)| k$.

Corollary II: Suppose $\forall X \subseteq A, \ |N(X)| \geq |X| - \Delta$ where Δ is some constant. Then there is a matching of size $|A| - \Delta$

Proof: Define a graph G' from G by adding Δ imaginary elements to B, then the Hall condition is satisfied and implies a perfect matching of A in G'. Deleting the edges incident to the imaginary elements a matching of size $|A| - \Delta$ remains. $\square$

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We will continue with max-min theorems today.

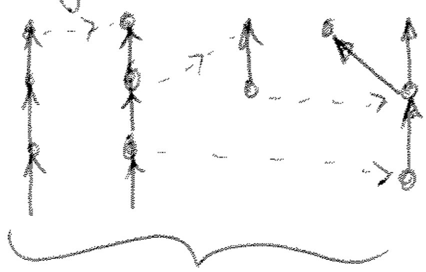
Thm.: (Gallai-Milgram)

Def.: A directed graph has edges that have one of two possible directions: $\bullet\!\to\!\bullet$ $\bullet\!\leftarrow\!\bullet$

The independence number of G, $\alpha(G)$, is defined to be the maximal size of an independent subset of $V(G)$.

The vertices of G can be covered by at most $\alpha(G)$ disjoint directed paths.

E.g.



$\alpha(G)$

(The minimum number of disjoint directed paths covering $V(G)$)

$\leq \max \# \alpha V(G)$

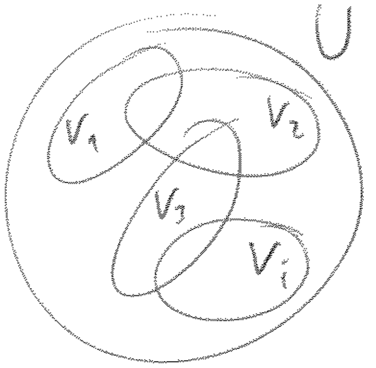
Def.: Partially ordered set (poset), $(X, <)$ is such that $\forall\ x, y, z \in X$

i) $x<y \wedge y<z \Rightarrow x<z$ (transitivity)

ii) $x<x$

iii) $x<y,\ y<x \Rightarrow x=y$

So for a directed graph G with $V(G)=X$, a poset, we define $x \to y$, $\overrightarrow{xy} \in E(G) \Leftrightarrow x<y$ but $x \neq y$.

E.g.: 

$V_1, V_2, \ldots \subseteq U$

Now we can assign to $V_1, V_2, \ldots$ a poset $X = \{V_1, V_2, \ldots\}$ where $V_i < V_j$ iff $V_i \subseteq V_j$.

The poset conditions clearly holds.

Def.: A chain is a sequence of elements of a poset $(X, <)$ e.g. $X_1 < X_2 < \ldots < X_k$ where all X_i are distinct.

An antichain is a set of elements of X so that no two of them are comparable. I.e. $X_1, X_2, \ldots, X_k$ s.t. $\nexists\, i,j,\ i \neq j$ s.t. $X_i < X_j$. For example disjoint sets can not be compared in the sense of the above example.

In a graph we can associate a directed path with a chain and independent vertices with an antichain.

Application of Gallai-Milgram theorem to this association gives: If $\alpha(G)$ = maximum size of an antichain then we can cover X by $\alpha(G)$ disjoint chains.

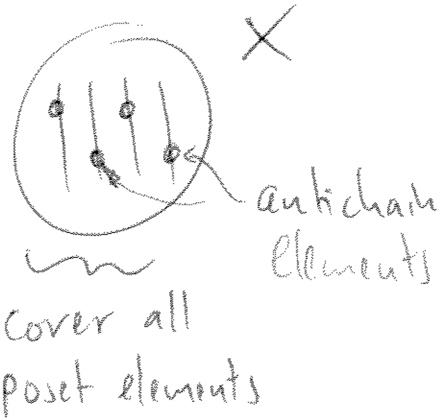
$\alpha(X, <) := \alpha(G)$ = maximum size of an antichain

$\beta(X, <) :=$ minimum number of chains that cover X

$$\beta(X, <) \leq \alpha(X, <)$$

But here it is impossible to have $\beta < \alpha$.

To see this suppose the contrary: A cover which has strictly less than α chains.

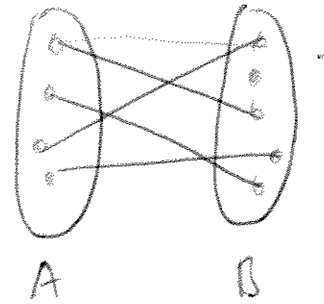


But no two dots can be on one line, else they would not belong to the same antichain.

$\square$

Corollary 2: (König's theorem) max matching = min vertex cover for any bipartite graph.

Proof: E.g.



Let all edges go from left to right (as we need to construct a directed graph). So a directed path is here either a vertex or one edge.

So the minimum number of disjoint directed paths is clearly equal to the maximum size of a matching, + some single vertices. I.e.

$$\text{min \# disjoint directed paths} = \text{max } \overbrace{\text{matching}}^{:=M} + (|A|+|B|-2\max M)$$

$$= |A|+|B| - \max M \leq \alpha(G)$$

which is smaller than $\alpha(G)$ by Gallai-Milgram. $\alpha(G)$ is the smallest number of independent vertices. For bipartite graphs, $\alpha(G) \underset{\substack{\uparrow \\ \text{lemma}}}{=} |A|+|B| - \text{min vertex cover}$.

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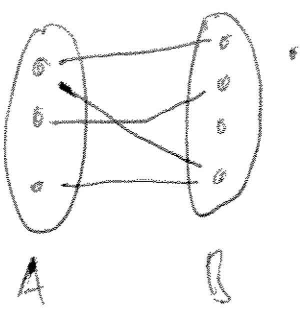
Assuming we believe the lemma, ~~and not~~

$|A| + |B| - \max M \leq |A| + |B| - \min$ vertex cover

$\Rightarrow$ $\max M \geq \min$ vertex cover, proving König by Gallai-Milgram. $\square$ (Since $\max M \leq \min$ vertex cover is trivial).

Lemma: Bipartite graph : $\alpha(G) = |A| + |B| - \min$ vertex cover

Proof: E.g.



. The complement of an independent set is a vertex cover in a bipartite graph.

Corollary I : (Dilworth theorem)

Let $(X, <)$ be a poset, then

max # antichain = min # chains that cover X.

Proof attempt for G.M. for undirected graphs.

List longest possible paths,

$\Big\} k$ disjoint paths

and take the initial vertex of each path, which must be an independent set by our choice of paths.

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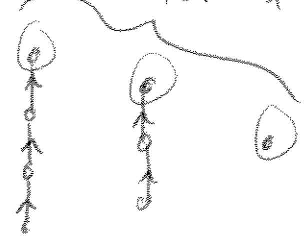
Proof: (Gallai - Milgram)

Let $\mathcal{P}_1, \mathcal{P}_2$ be two directed path covers of $V(G)$, i.e. sets of paths that cover $V(G)$. (Note, $\circ$, $\bullet\!\!-\!\!\circ$, $\bullet\!\!-\!\!\bullet\!\!-\!\!\circ$, are all paths here). Define a partial order:

$\mathcal{P}_1 < \mathcal{P}_2 \iff |\mathcal{P}_1| < |\mathcal{P}_2|$, and ter $\mathcal{P}_1$:= terminal vertices

We have directed paths, $\mathcal{P}_1$:

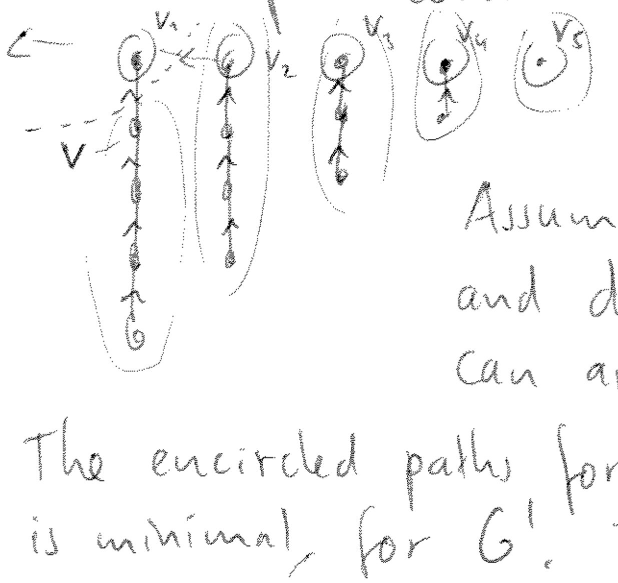


$\mathcal{P}_1 < \mathcal{P}_2 \iff |\mathcal{P}_1| < |\mathcal{P}_2|$ and $\text{ter}\,\mathcal{P}_1 \subset \text{ter}\,\mathcal{P}_2$.

We prove by induction on $|V(G)|$ $(= |G|)$ the following statement:

(*) If $\mathcal{P}$ is a minimal path cover with respect to $<$ then we can pick one element from each path of $\mathcal{P}$ that form an independent set.

Suppose (*) is true $\forall\ k \leq |V(G)|$. Pick some minimal path cover $\mathcal{P}$, e.g.



with terminal vertices v_i.

Recall that $\mathcal{P}$ is minimal.

Assume, w.l.o.g. that $\overrightarrow{v_2 v_1} \in E(G)$, and define $G' := G - v_1$ where one can apply the induction hypothesis.

The encircled paths form clearly a path cover, which is minimal, for G'. This can be shown by contradiction.

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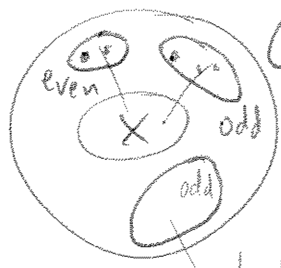
Suppose the contrary, i.e. there is a path cover $\mathcal{P}'$ of G' where the terminal vertices form a proper subset of $\{v_1, v_2, v_3, \dots, v_k\}$ $\rightarrow$ this path cover contains $\leq k-1$ paths. v must not belong to ter $\mathcal{P}'$. I.e. $v_1, v_2 \notin$ ter $\mathcal{P}'$, else we have a contradiction with the assumption that $\mathcal{P}$ is a minimal cover. Thus $|\text{ter } \mathcal{P}'| \leq k-2$ ↯, since one could add the path just consisting of v_1 to obtain a cover of G. So we started with a minimal path cover, we looked at the set of terminal vertices which, in the case that they are not independent, one can show by induction that (*) holds. $\square$

Thm: (without proof) (Tutte) ($\Rightarrow$ Hall's theorem)

Def.: A perfect matching in a graph means that every vertex is covered by a matching. This is also called 1-factor: E.g.

A graph G contains a perfect matching (1-factor) iff $\forall X \subseteq V(G)$, ($G-X$ meaning that all vertices and edges of X are removed from G) ~~so e.g.~~ is such that the number of odd connected components of $G-X \leq |X|$.

E.g.



If $\exists$ a perfect matching, each odd component must have an edge into X. The result follows.

at least one vertex cannot be perfectly matched in this component

Def.: A graph is said to be k-connected if there is no set of $k-1$ vertices whose removal disconnects the graph and $|V(G)| > k$.

E.g. if a graph is 2-connected then the following hold:

→ connected

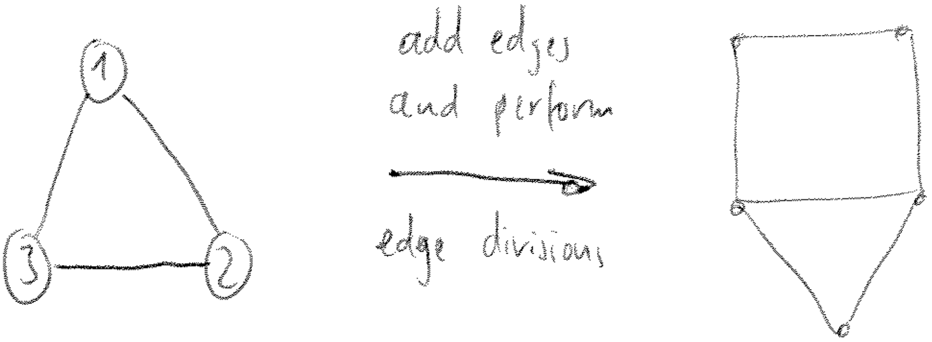
→ $\min d(v) \geq 2$

→ any cycle is 2-connected

→ Subdividing a graph which is 2-connected renders it 2-connected

→ every vertex pair lies on a cycle

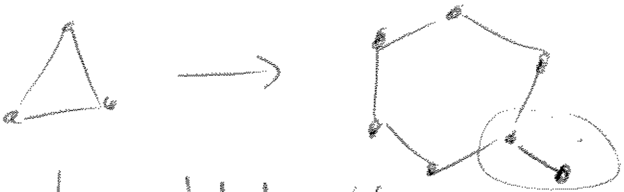
→ if G is 2-connected, then $E(G) + e$ is also 2-connected.



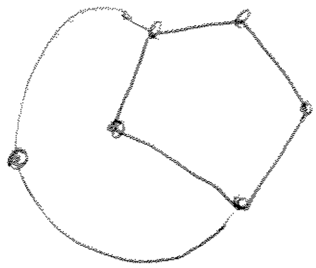
Not every two connected graph has all vertices on a cycle:

Proposition: A graph G is 2-connected iff it can be obtained from a triangle using only edge additions and subdivisions.

Proof: "⇒" To show: Any 2-connected graph can be obtained from a triangle, allowing only edge addition and subdivision.
Let G be any 2-connected graph, then there must be some cycle. This cycle can readily

obtained via subdivision. E.g. 

Every edge within the cycle can be added. Now to get a connection to the remaining graph, one can use the fact that every vertex pair in a 2-connected graph must lie on a cycle.



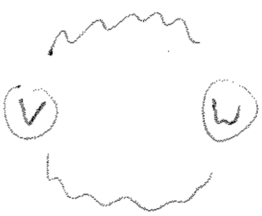
We can continue this procedure until we have the complete graph. □

Thm.: G is k-connected iff any pair of vertices has k disjoint paths. (Menger)

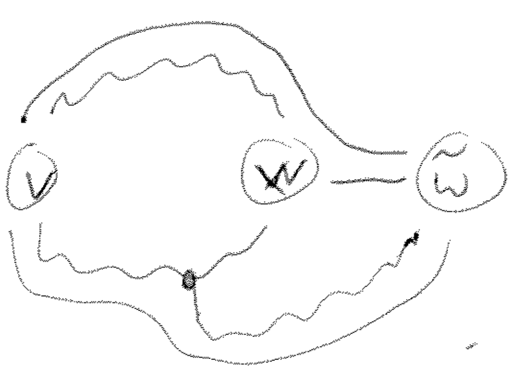
Proof.: To ~~proot~~ prove the theorem for $k = 2$, use induction on the distance of two vertices.

Base case: distance = 1, obvious.

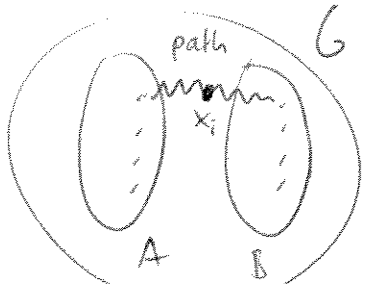
Inductive step: $\mathrm{dist}(v,w) = t$ and v,w lie on a cycle.



$\tilde{w}$, $\tilde{w}$ must be connected to v, so



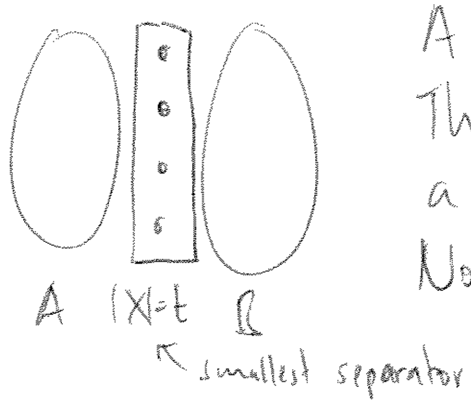
Now let us consider the general case. Let's prove a slightly different statement first. Let G be any graph, $A, B \subseteq V(G)$ and let $X \subseteq V(G)$ contains vertices sitting on any path that go from A to B.



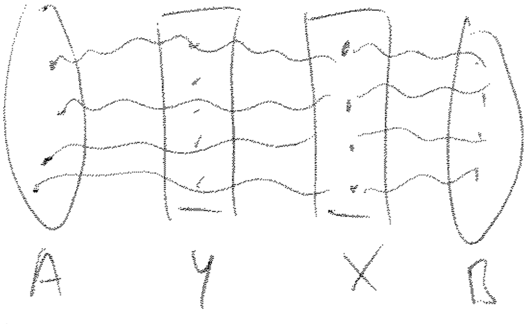
So X is the smallest set of vertices that blocks all paths from A to B.

Claim: max # disjoint path $A \leftrightarrow B$ $= |X| =$ size of smallest set that blocks all paths from A to B.

"$\leq$" is easy to see. Harder is "$\geq$" which we'll prove by induction on # edges in G. Consider paths between

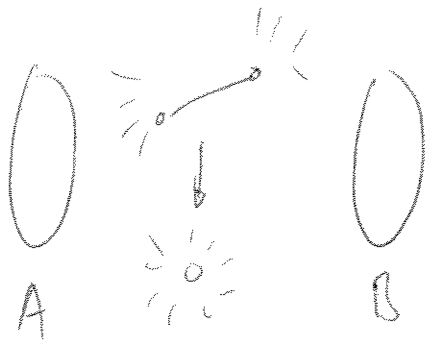


A and X. (paths are disjoint!) Then we can again say that there is a set Y, separating A and X. Now $|Y| \geq t$ as $|X|$ is minimal.



One can always find a separator X such that between two x_i there is an edge. (We need such an edge to apply induction.)

An edge contraction:  .



$t-1$ vertices that block paths.

Going back to the initial graph, we can prove the claim. □

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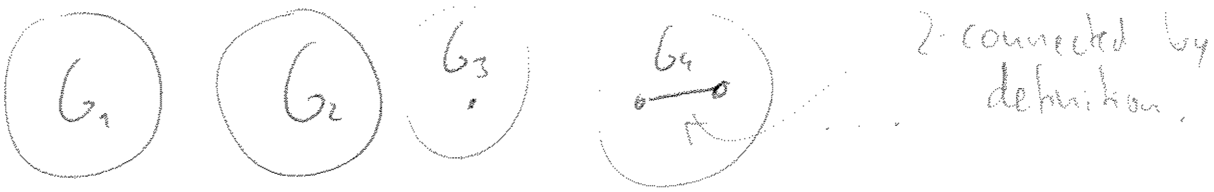
Def.: A graph is k-connected if $|V(G)| > k$ and the removal of any set of k-1 vertices does not disconnect the graph.

Thm.: (Ear-Decomposition) A graph is 2-connected iff it is constructible via "ear"-additions.

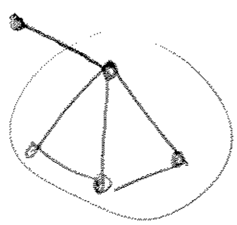
If G is not 2-connected, it falls into connected components, say $G_1, G_2, \dots$ which are mutually disjoint.



Now we look at one connected component.

Def.: A <u>block</u> of a graph G is a maximal subgraph of G that is 2-connected. Where maximal means that no edge, and no vertex can be added to it, so that it remains 2-connected.

E.g.: ← Cannot be included as it violated 2-connectivity.



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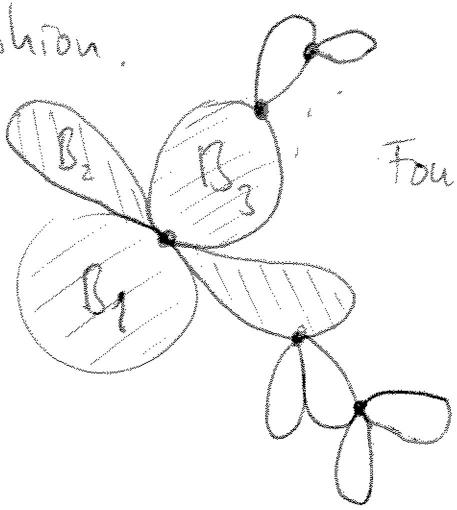
Thm.: (Cactus Decomposition of Connected Graphs)

Let $B_1, B_2, \ldots$ denote the blocks of a graph G:

i) $|V(B_i) \cap V(B_j)| \leq 1 \quad \forall \quad i \neq j$

ii) The blocks are connected in a tree (cactus) like fashion.

E.g.:

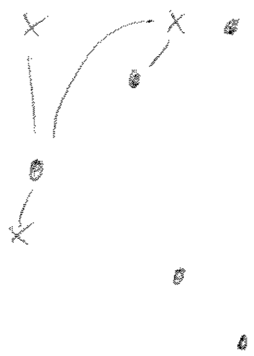


Four blocks may meet at this vertex.

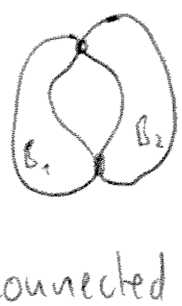
Consider the vertices contained in at least **two** blocks are drawn below again.

Assign a vertex "$\times$" to each block, and connect the crosses with the dots if the dot is included in the blocks $\times$.

This will always be a bipartite tree



Proof: i) By contradiction. Assume there are two blocks that intersect in more than one point. But $B_1 \cup B_2$ is also 2-connected, contradicting maximality of the blocks. ⚡



2-connectedness of $B_1 \cup B_2$ follows again by contradiction. Assume one deletes some $x \in B_1 \cup B_2$ and $B_1 \cup B_2$ is disconnected. ⚡

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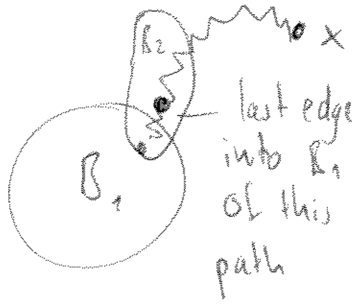
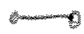
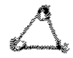
i) => ii)



The graph is connected, so there must be a graph connecting any x outside B_1 to B_1.

~~Either~~ There is a maximal 2-connected subgraph that contains this last edge.

Note  is 2-connected and  3-connected by definition.

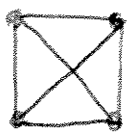
$\square$

Informally, connected components (into which every graph can be split) which one can split each into blocks, for each block the ear-thm. follows.

Planar Graphs

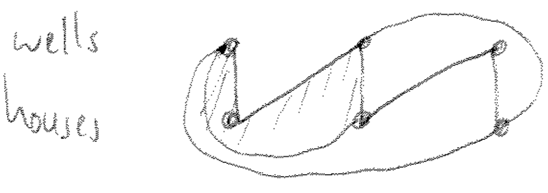
Note there is a slight difference between "planar" graph and "plane" graph. In both meanings, the graph can be drawn without crossing edges. Plane graphs are particular graphs without crossing.

Def.: noncrossing drawing possible => planar graph, e.g.

 is planar as it can be redrawn as 

but the two are different plane graphs.

E.g.: 3-houses 3-wells puzzle: Impossible to find non-crossing roads from each house to each well.



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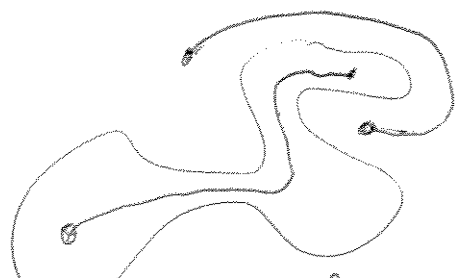
Thm.: (Jordan-Curve-Theorem)

Any non-self-intersecting closed curve γ divides the plane into two regions A, B s.t.:

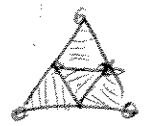
i) any two points in the same region can be connected by a continuous curve non-intersecting γ,

ii) Any curve connecting any 2 points of different regions must intersect γ

E.g.:



γ



5 faces.

Proposition: (3-house-3-wells)

$K_{3,3}$ = Complete bipartite graph with 3-3 vertices in its parts. $K_{3,3}$ is not planar.

(Proposition: (Kuratowski's thm.) K_5 is not planar. later)

Thm.: Let G be a plane graph, connected. There are $|F(G)|$ faces of G. (The exterior is also a face!).

$$|V| - |E| + |F| = 2.$$

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E.g.: For a tree, $|E| = |V| - 1$, $|F| - 1$.

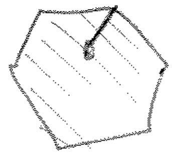
Proof: By induction on $|F|$.

Base: $|F| = 1 \Rightarrow G$ must be a tree, as it must be connected and acyclic. The Jordan-Curve theorem assures acyclicity.

Hypothesis: Assume $|V| - |E| + |F| = 2$ $\forall$ connected plane graphs with $< |F|$ faces.

$|F| \geq 2 \rightarrow G$ has at least one cycle, as the boundary of the face is ~~the~~ a cycle.

E.g.:



Now remove one edge of the face boundary, so # faces will be reduced by one.

$|F'| = |F| - 1$ and $|E'| = |E| - 1$

but $|V'| = |V|$.

Then $|V| - |E| + |F| = 2$ implies

$$|V'| - |E'| - 1 + |F'| + 1 = 2$$

$$\Rightarrow |V'| - |E'| + |F'| = 2 . \quad \square$$

Corollary: Every planar graph with at least 3 vertices n, has at most $3n - 6$ edges. I.e.

$|E| \leq 3|V| - 6$.

This will imply K_5 is not planar, as it violates this inequality: $10 \leq 3 \cdot 5 - 6$ ↯ But it does not fail for $K_{3,3}$!

Characterisation of Planar Graphs

Corollary: Let $|V| \geq 3$ then $|E| \leq 3|V| - 6$.

Proof: Let a maximal plane graph be such that every face is a triangle and every edge must be incident to two faces. Let us transform G into $\tilde{G}$, maximal. Let us have faces $f_1, \dots, f_\ell$ with sizes $d(f_i)$.
Now $\sum d(f_i) = 2m' = 3\ell = 3(2 + m' - n)$, $m \leq m' = 3n-6$
We only added edges to maximize G. $\square$

$\Rightarrow$ K_5 is not planar! ($m = 10$, $n = 5$ $\Rightarrow$ $m \overset{\not}{\leq} 3n - 6$)

Exercise: If G is bipartite and planar $\Rightarrow$ $m \leq 2n - 4$.
$K_{3,3}$ can thus be not planar, since $K_{3,3}$ has $m = 9$, $n = 6$.

Corollary: Every graph containing $K_{3,3}$ or K_5 or both cannot be planar.

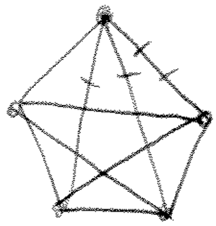
Def.: Let X, Y be graphs. X is said to be a topological
(19) minor of Y if Y contains a subdivision of X as a subgraph. I.e. the edges of X are replaced by independent paths.

graph 28/03 2

E.g. C_4, the square, is a topological minor of

Y :

Proposition: Assume K_5 is a topological minor of a graph G. Then G is non-planar.

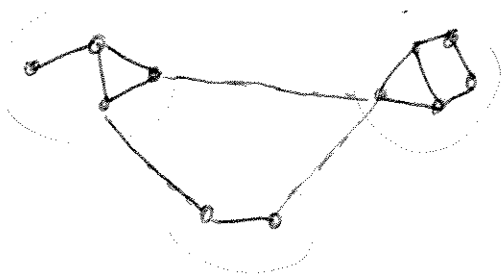


So every graph including K_5 cannot be planar.

Def.: X is said to be a minor of Y if we can obtain X starting from Y after a series of edge contractions and edge-deletions or vertex-deletions.

Remark: Every topological minor is also a minor.

E.g.: G



Every connected component can be transformed into a single vertex by the allowed operations, so $\triangle = K_3$ remains.

Note that G also has $\triangle$ as a topological minor.

Def.: Vertices contracted to a single vertex are said to be branch sets and vertices spanning TX to be branch vertices.

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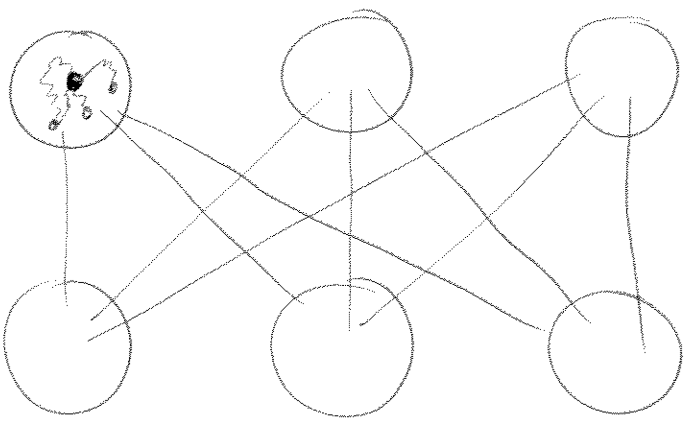
Proposition: A plane graph does not contain TK_5, $TK_{3,3}$, MK_5 nor $MK_{3,3}$.

Proof: Trivial.

Proposition: A graph contains TK_5 or $TK_{3,3}$ iff it contains MK_5 or $MK_{3,3}$ ~~respectively~~.

Proof: "$\Rightarrow$" Trivial.

"$\Leftarrow$" $MK_{3,3} = TK_{3,3}$ is to show:

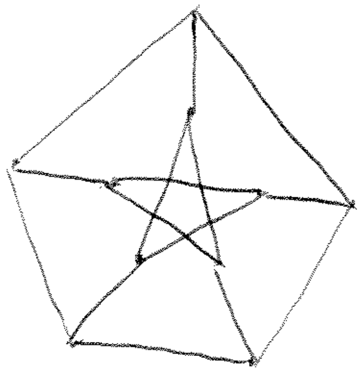


For each connected component, a representative can be selected, being connected to three other components.

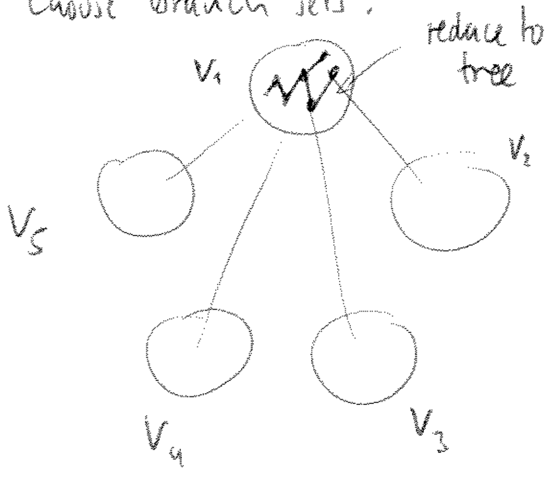
$\leftarrow$ branch set

$MK_5 \Rightarrow TK_5$ is to show:

Consider the Pertersen graph, it contains no TK_5, but MK_5.



Choose branch sets:



Let the smallest $MK_5 = K$.

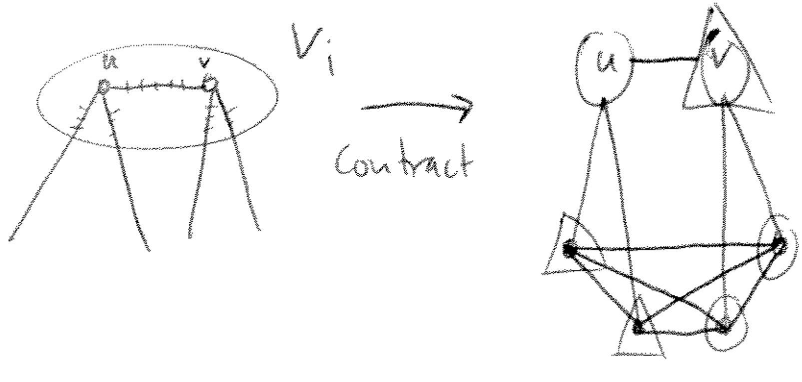
$K(V_i) =$ "a tree $\forall i$".

$T_i :=$ "add four edges to $K(V_i)$"

graph 28/03 4

T_i has exactly 4 leaves in other branch sets if all T_i's are $TK_{1,4} \Rightarrow TK_5 \subseteq K$.

Otherwise



to retain $MK_{3,3}$.

Hence we showed $MK_5 \Rightarrow MK_{3,3} \Rightarrow TK_{3,3}$. $\square$

Thm.: (Kuratowski, Wagner)
The following statements are equivalent for a graph G:
1) G is planar
2) G contains neither K_5 nor $K_{3,3}$ as a minor
3) G contains neither K_5 nor $K_{3,3}$ as a topological minor.

Proof: We saw already today $(1) \Rightarrow (2)$, $(1) \Rightarrow (3)$ and $(2) \Leftrightarrow (3)$.
It remains to show that $(2) \Rightarrow (1)$, which we'll show in the following steps:
Step 1: If G is 3-connected without MK_5 and $MK_{3,3}$, then G is planar.
Proof: Induction on $|G|$. $|G| = 4$, $G = K_4$ ✓.
Hypothesis: It hold for $|G| > 4$. Here we need a lemma: There is an edge in any 3-connected graph s.t. G/e is 3-connected.

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let us choose e s.t. $e = xy$ and G/e is planar

$v_{xy} \in V(G/e)$ $\qquad$ $xy \longrightarrow v_{xy}$. $\qquad$ $G - e$ = deletion

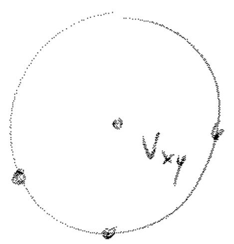
contraction

$G/e - v_{xy}$ becomes 2-connected.

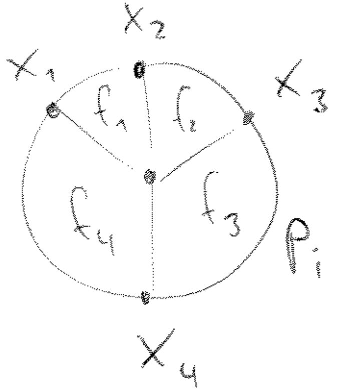
A planar 2-connected graph has all faces surrounded by cycles. Let $X = N(x) - y$ and $Y := N(y) - x$,

$X \cup Y \subseteq C$

$G' - \{ \overbrace{v_{xy} v}^{\text{edge}} \mid v \in Y \setminus X \}$



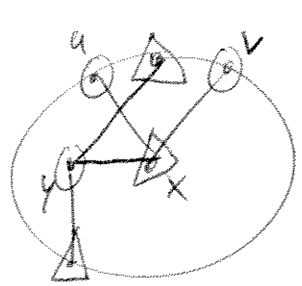
Drawing of $G - x$ with $v_{xy} \to x$.
We almost succeeded as we want to draw G. How to place $\cancel{x}$ y?



If some P_i contains Y then we put y in f_i.

Otherwise:

This cannot happen, since $TK_{3,3}$ is not allowed!



9

Two edges by two-connectedness, three edges at x as 3-connected.

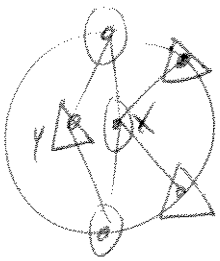
Case 1: y has all neighbors in one P_i

Case 2: y has one neighbor not on a path

Case 3: y and x can have at most two common neighbors. Else TK_3

graph 28/03 6

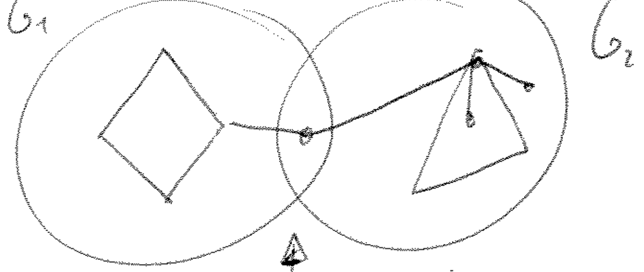
So in this case we rediscover $TK_{3,3}$, which is not allowed.



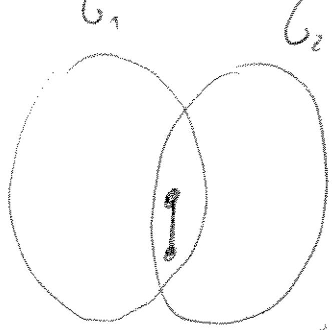
This completes step 1.

Step 2: If G is edge-maximal without TK_5 and $TK_{3,3}$ and with $\kappa(G) \leq 2$ and $G = G_1 \cup G_2$, where G_1, G_2 are proper and induced subgraph, $|G_1 \cap G_2| = \kappa(G)$. (Induced means we have to include all edges connected to the vertices chosen). Then $G_1 \cap G_2 = K_2$ and G_1, G_2 are also edge maximal without TK_5 or $TK_{3,3}$.

Proof:



cannot be like this, it must be



It will follow in detail next time.

Step 3: If G is edge-maximal without TK_5 and $TK_{3,3}$ and $|G| \geq 4$ then G is 3-connected.

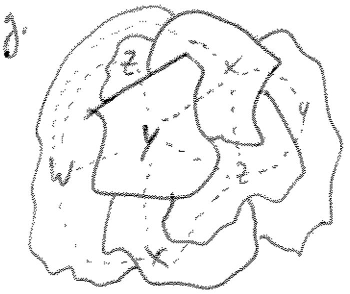
(Start with any graph. If it is not 3-connected one can add edges until maximal).

Proof: Next time. $\square$

Coloring Graphs

Today we start with vertex colorings. It comes from map-producers. Let the countries ~~be~~ connected (simply), i.e. no holes in it and let no point belong to more than four countries.

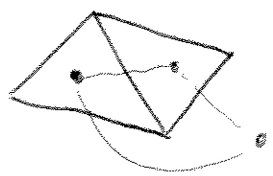
E.g.



Proposition: Given these assumptions, every map can be drawn with 4 colours, i.e. no two neighboring countries have the same colour.

Proof? Pick one point in each country and connect it to (other capitals) as sketched. This gives rise to a planar graph. Hence the proposition can be reformulated:
Every planar graph is 4-vertex colorable.
(Appel-Haken) There is a computer-assisted proof.

Def.: For any plane graph we can construct it's dual:
(No parallel edges, i.e.)



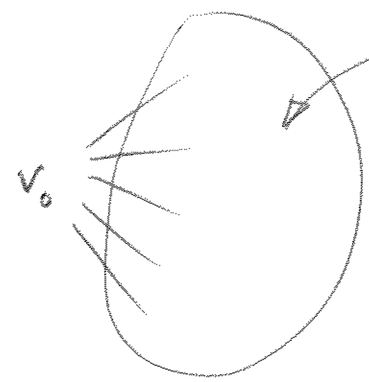
Thm.: (Six-Colour-Thm.) 6 colors suffice for all planar graphs.

Proof: $|E| \le 3|V| - 6 \Rightarrow \sum_{v \in V} d(v) = 2|E| \le 6|V| - 12 < 6|V|$

$\Rightarrow \frac{1}{|V|} \sum_{v \in V} d(v) < 6$ avarage degree.

Pigeonhole implies: $\exists\ v_0 \in V$ s.t. $d(v_0) \leq 5$.
Let's proceed by induction of $|V|$. Clearly it holds for $|V| \leq 6$. Assume G has more than 6 vertices. There is at least one vertex with degree ≤ 5, so consider

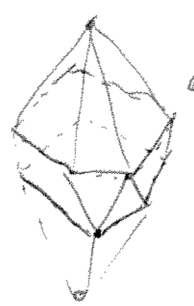
$G - v_0$



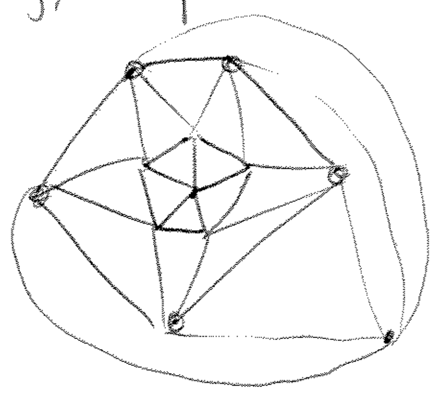
6 colorable by inductive hypothesis.
As v_0 has at most 5 neighbors.

$\square$

We cannot apply the same strategy to prove the 5-colour conjecture.



two pentagons and two extra points $\Rightarrow$



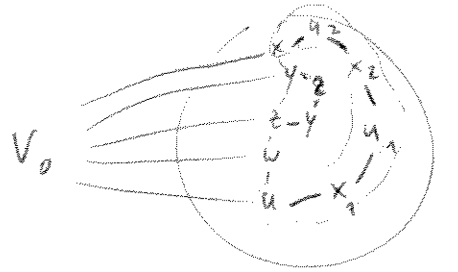
a 5 regular graph on 12 vertices.

Thm.: (Five-Colour-Thm.) 5 colors suffice for every planar graph.

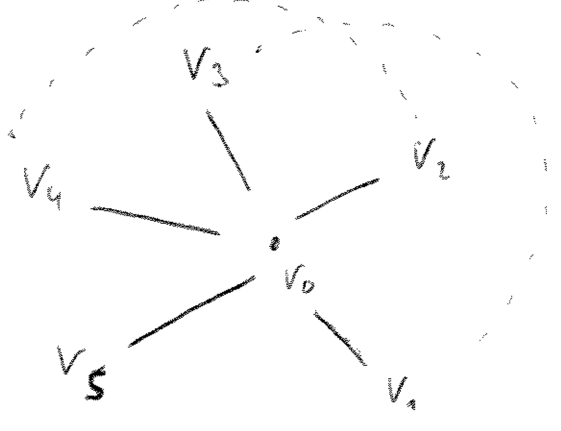
Proof: By induction on $|V|$. Clearly, $|V| \leq 5 \Rightarrow \checkmark$.
Inductive hypothesis: Statement true for $< |V|$ vertices.
If there exists $v_0 \in V(G)$ s.t. $d(v_0) \leq 4$ then we can apply the same procedure as in the 6-color thm. proof.

By the argument $\frac{\sum d(v)}{|V|} < 6$, $\min_{v \in V(G)} d(v) = 5$.

Pick v_0 with $d(v_0) = 5$ and consider $G - v_0$.



If all neighbors of v_0 have different colours, we have a problem.
Take the two colours x and u and note that there must be a path (G is planar $\Rightarrow$ connected) that alternates between colour x and u. This holds because of the following:
Let V^* denote the set of all vertices that can be reached from u by an alternating path of the sketched form.
If x is not contained in V^*, then switch colours in V^*, i.e. recolour. But this means u and x will have the same colour. Similarly, w and y can be connected by a path that has alternatively colours w and y.[2] ~~No since~~



Impossible to connect v_1 and v_3 by an alternating path P_{13} and also connect v_2 and v_4 by an alternating path. So two neighbors must have the same colour
$\Rightarrow$ 5 colours suffice. $\square$

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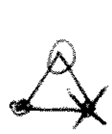
Why does this proof not work for the 4-colour conjecture?

Def.: The chromatic number of a graph G is the smallest k such that the vertices of G can be coloured by k colours such that no two adjacent vertices get the same colour.

cromatic number of G = $\chi(G)$

E.g.:  $\chi(G) = 2$;  $\chi(C_k) = \begin{cases} 2 & \text{if } k \text{ is even} \\ 3 & \text{if } k \text{ is odd} \end{cases}$

$\chi(K_k) = k$. 

Def.: Denote the clique number of G by $\omega(G)$, where $\omega(G) = |\max K_k \subseteq G|$.

E.g.: $\omega(C_k) = 2$ if $k \neq 3$.

Thm.: $\forall$ graphs G, $\chi(G) \geq \omega(G)$.

Perhaps $\chi(G) \leq f(\omega(G))$ for some function f? No.

Proposition: There exist Δ-free graphs ($\omega(G) = 2$) with arbitrarily large cromatic number.

So it is difficult to bound $\chi(G)$ from above.

Thm.: (Erdös) For every k, ℓ $\exists$ a graph $G_{k,\ell}$ which has no cycle of length $\leq k$ but $\chi(G_{k,\ell}) \geq \ell$.

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So this theorem is stronger than the previous proposition, which is the theorem for $k=3$.

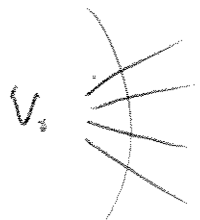
Def.: The girth of a graph is the length of the shortest cycle in G.

Def.: Let $\Delta(G) = \max_{v \in V(G)} d(v)$.

Thm.: $\chi(G) \leq \Delta(G) + 1 \quad \forall$ graph G.

Proof: By induction on $|V(G)|$. $|V| \leq \Delta(G)+1$ ✓

There is a ~~coulo~~ colour availible for v_0.



$\leq \Delta$

$\square$

Brooks's theorem.

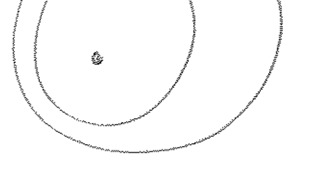
Thm.: $\chi(G) \leq \Delta(G) \quad \forall$ connected graph that is not complete and not an odd cycle.

Proof: (Next time)

Remark: There are no good estimates for $\chi(G)$ as this is computationally hard.

Def.: Let $\delta(H) = \min_{v \in H} d(v)$, and $\mathrm{col}(G) = \max_{H \subseteq G} \delta(H) + 1$

Thm.: $\chi(G) \leq \mathrm{col}(G) \quad \forall G$.



$\Rightarrow \chi(G) \leq \Delta(G) + 1$

graph 11/04 1

Proposition: $\chi(G) \geq \omega(G) = \max\{d \mid K_d \subseteq G\}$
chromatic nb. clique nb.

Proof: Trivial.

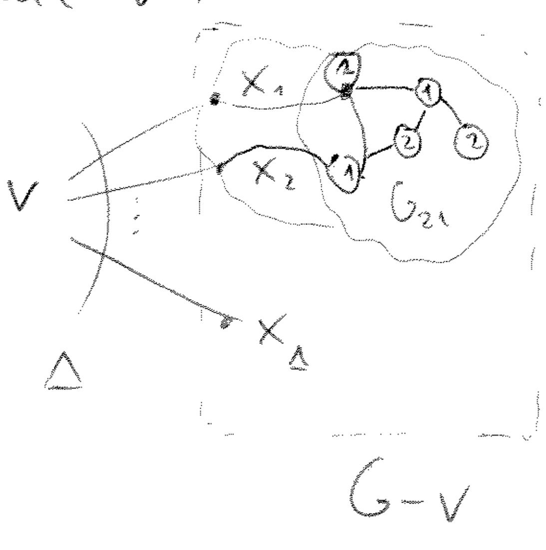
Proposition: $\chi(G) \leq \Delta(G) + 1$ $\forall$ graphs G.

Proof: (Last time). $\square$

The last proposition cannot be improved, i.e. it is tight for $K_{\Delta+1}$.

Thm.: (Brooks) $\chi(G) \leq \Delta(G)$ $\forall$ connected G other than $K_{\Delta+1}$ and odd cycles.

Proof: Assume $\Delta > 2$. Apply induction on $|V(G)|$. Consider the following minimal counterexample. Here $d(v) = \Delta$, a fixed Δ. (It is to show that the graph can be coloured by Δ colors.) The only exception where we couldn't use induction is when we had a complete graph $K_{\Delta+1}$ left, forbidden. So $d(v) = \Delta$ can be assumed $\forall$ $v \in V(G)$. This is the only case we have to analyse. Pick some v and assume all neighbors have different colors.



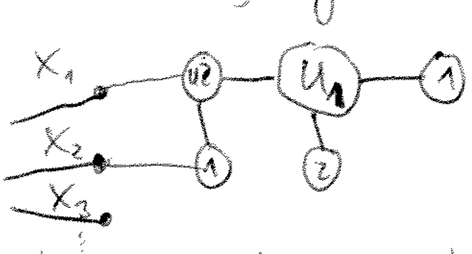
This cannot happen as the following contradiction arises.

Let the color of x_i be i. For every $i \neq j$ consider the subgraph of $G-v$, G_{ij}, consisting of all vertices of ~~degree~~ color i or j. I.e. the subgraph only having two types of vertices. Consider the following statements:

i) In G_{ij} the vertices x_i and x_j are in the same connected component, because else the vertices that can be reached from x_i by an alternating ij-path would be a connected component of G_{ij} that contains x_i and we could switch colours, and so we have an unused color left for v. (Same argument as in 5 color thm..)

ii) The component of G_{ij} that contains x_i and x_j is a simple alternating path P_{ij}.

If P_{ij} is not the complete connected component including X_i and X_j, there must be a vertex along the path which is the first to be connected to a vertex of G_{ij} not belonging to P_{ij}. E.g.:



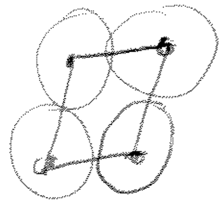
But if u_2 has three neighbors of the same color, here 1, and as each vertex has degree Δ, one color is not used at all, $\Rightarrow$ u_2 could be recolored, contradicting i). Thus ii) must hold.

iii) P_{ij} and P_{jk}, where i,j,k distinct, are such that

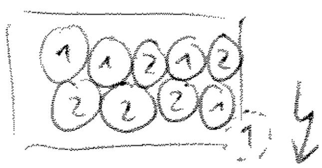
$$P_{ij} \cap P_{jk} = \{x_j\}.$$

graph 11/04 3

Coin Puzzles: i) Place coins on a table, they cannot overlap, then any system of non-overlapping equally sized coins can be coloured by 4 colours such that no two incident coins have the same color. (Don't use 4-colour theorem!)



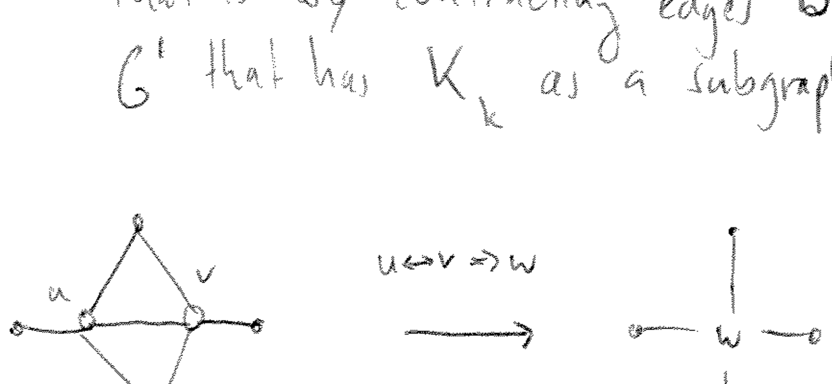
ii) Two people are putting down coins of same size on a table of finite size, coins mustn't overlap. The one who cannot put down a coin, looses.



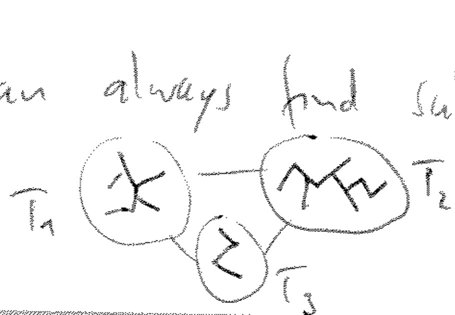
Prove that the player who places the first coin can always win.

Conjecture: (Hadwiger) If a graph G has $\chi(G) \geq k$ then G contains K_k as a minor.
That is by contracting edges of G we can obtain G' that has K_k as a subgraph. (Trivial for $k=2$).

E.g.:



So one can always find subtrees T_i that can be connected:



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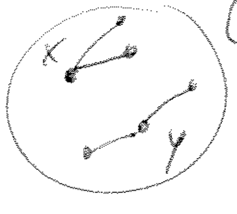
The conjecture is easy for $k=3$, as then it must contain an odd cycle which can be contracted to a triangle.

Thm.: (Hajós) Characterisation of G with $\chi(G) \geq k$.

Def.: k-constructible graphs, (let k be fixed) satisfy:

i) K_k is k-constructible, e.g.  K_3.

ii) Let G be k-constructible and assume it has two vertices that are not connected, say $xy \notin E(G)$, $x,y \in V(G)$. E.g.:

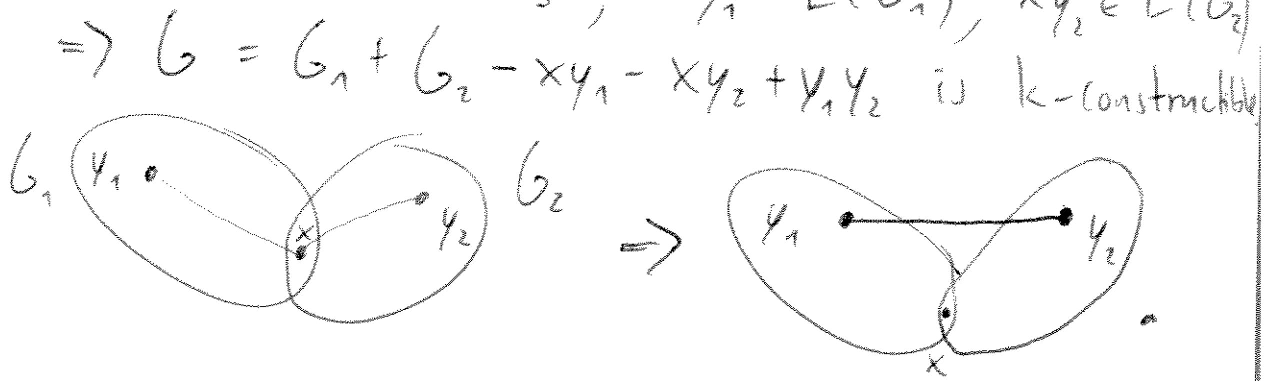


Add xy to G and then contract this edge, i.e. $(G+xy)/xy := G'$,

Then $G+xy \,/\, xy$ is also k-constructible.

iii) Suppose G_1 and G_2 are k-constructible graphs, $V(G_1) \cap V(G_2) = \{x\}$, $xy_1 \in E(G_1)$, $xy_2 \in E(G_2)$

$\Rightarrow G = G_1 + G_2 - xy_1 - xy_2 + y_1 y_2$ is k-constructible



G

After this definition, Hajós theorem can be stated.

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(Hajós) $\chi(G) \geq k \iff G$ contains a k-constructible subgraph.

Proof: "$\Leftarrow$" All k-constructible graphs have $\chi(G) \geq k$.

By induction: $\chi(K_k) \geq k$ clearly holds, so i) ✓.
It remains to show ii) doesn't create a graph with $\chi(G') < k$. Also iii):

$\chi(G_1) \geq k$ and $\chi(G_2) \geq k \Rightarrow \chi(G') \geq k$. Since assume not and look at the definition: If

$\chi(G'_m) < k$, so you have a good coloring of G_2 so $\chi(G_2) < k$, contradiction!

"$\Rightarrow$" $\chi(G) \geq k \Rightarrow G$ contains a k-constructible subgraph.

Next time.

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(The three steps can be used to construct k-constructable graphs)

Def.: k-constructible gaphs satisfy the following:

i) K_k is k-constructible

ii) G k-constructible, $xy \notin E(G)$, $\Rightarrow (G+xy)/xy$ is k-constructible

iii) If G_1, G_2 are k-constructible then if $V(G_1) \cap V(G_2) = \{x\}$ and $xy_1 \in E(G_1)$ and $xy_2 \in E(G_2)$ then $G' := G_1 + G_1 - xy_1 - xy_2 + y_1y_2$ is k-constructible.

Thm.: (Hajós) $\chi(G) \geq k$ iff G has a k-constructible subgraph.

Proof: "$\Leftarrow$" Whichever graph we construct in the i), ii), iii)-way, it's chromatic number will be at least k. (The second property follows by contradiction, so does iii).)

"$\Rightarrow$" To show: If $\chi(G) \geq k$ then G has a k-constructible sub-graph. First recall equivalence relations:

A set X equipped with a binary relation $\sim$: $(X, \sim)$ satisfying:

i) $\forall x \in X \quad x \sim x$

ii) $\forall x, y \in X \quad x \sim y \Leftrightarrow y \sim x$

iii) $\forall x, y, z \in X, \quad x \sim y \wedge y \sim z \Rightarrow x \sim z$.

Equivalence classes partition X.

By contradiction. Consider the smallest counter example with respect to $|V(G)|$. If possible, add as many

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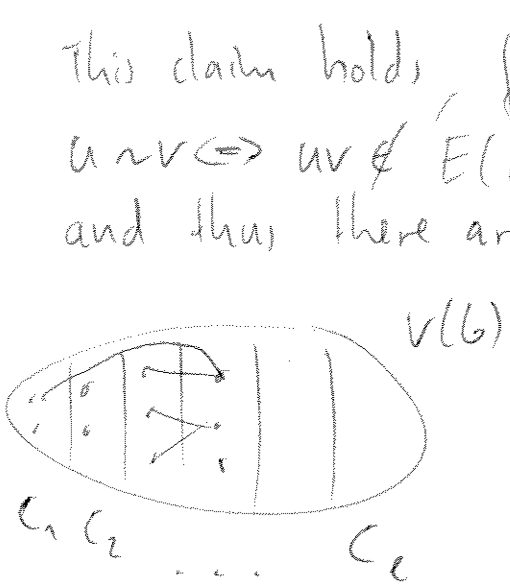
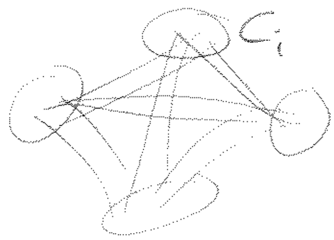
extra edges as possible, without creating any k-constructible subgraph. As $\chi(G) \geq k$, $|V(G)| \geq k$. Let this new graph with maximal # edges and minimal # vertices be G. We want to violate the minimality condition.

Claim: $\exists\ x, y_1, y_2 \in V(G)$ s.t. $xy_1, xy_2 \notin E(G)$ but $y_1y_2 \in E(G)$.

This claim holds, for if there are no such vertices then $u \sim v \Leftrightarrow uv \notin E(G)$ is an equivalence relation on $V(G)$, and thus there are ℓ equivalence classes. So G is a complete ℓ-partite graph. For each class one needs a different color. $\Rightarrow \chi(G) = \ell$.



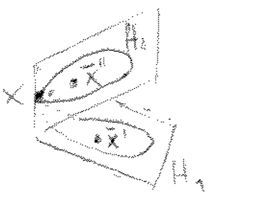


But $\chi(G) \geq k \Rightarrow \ell \geq k$ and G clearly contains K_ℓ as a subgraph thus it has a k-constructible subgraph. Thus G was no counterexample.

So the claim holds.

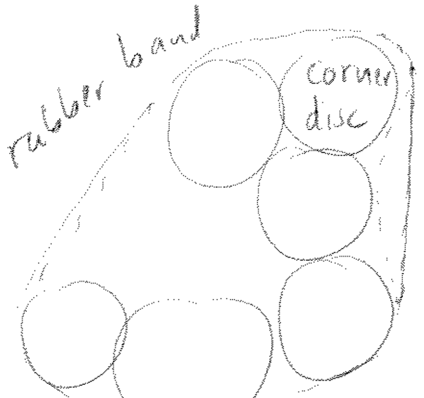
Adding the edge y_1x to G would result in a k-constructible subgraph H_1 and if adding y_2x would ... in H_2. Now suppose first that $V(H_1) \cap V(H_2) = \{x\}$, then iii) implies that $H_1 + H_2 - xy_1 - xy_2 + y_1y_2$ is k-constructible a subgraph of G! So G was not a counterexample.

Hence suppose that H_1 and H_2 have more than one point in common, i.e. $|H_1 \cap H_2| > 1$.



$\bar{x}'' \in H_2$ and $\bar{x}' \in H_1$. Their union is k-constructible by iii) but it is not a subgraph of G (as we duplicated $\bar{x}$). But rule ii) implies that when we glue together $\bar{x}'$ and $\bar{x}''$ we get a k-constructible graph. When eliminating all those double points (by glueing) we have a k-constructible subgraph of G. Again a contradiction to the assumption that there is a minimal counterexample. -$\times$-

Coin Problem: 4-colour theorem for touching coins:



By induction on # coins. Clear for $n \leq 4$. Now assume it holds for n. Consider the convex hull of the union of the coins. Call a disc a corner disc if the rubber bends around it (not only touching in one point).

Claim: Every corner disc touches at most 3 others.

Thus color each corner disc with a color and its three neighbors differently. This only works as the # coins is finite. The theorem also holds for infinitely many coins. What's the argument here?

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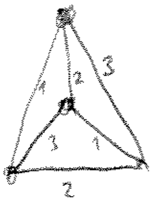
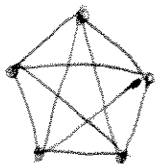
Colouring Edges

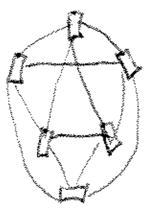
Def.: An edge-coloring is called proper if no two edges that share a vertex have the same color. Let the minimum number of colors for a proper edges coloring of G be its cromatic index $\chi'(G)$.

E.g.: G  $\chi'(K_3) = 3$

K_4  $\Delta(G) \leq \chi'(G)$; $\chi'(K_4) = 3$

K_5 

Def.: The line graph of G, call it H_G, is such that $V(H_G) = E(G)$ and $e, e' \in E(G) = V(H_G)$ are connected by an edge in H_G iff e, e' share a vertex.

E.g.: H_{K_4}:  $\forall\ G,\ \chi'(G) = \chi(H_G)$.

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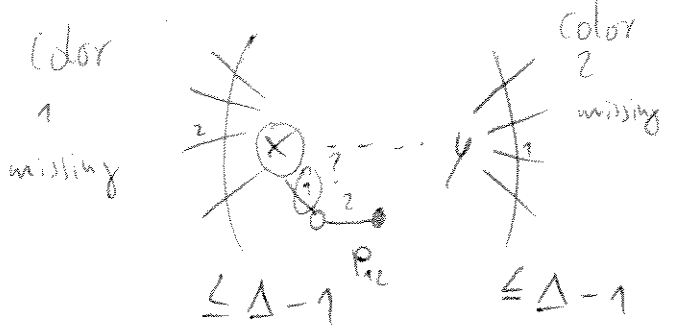
Thm.: (Vizing) $\chi'(G) \leq \Delta(G)+1 \quad \forall\, G$.
(tight for odd cycles).

Thm.: (König's theorem) For bipartite graphs, $\chi'(G) = \Delta(G)$.

Proof: By induction on $|E(G)| = m$. Pick an edge $xy \in E(G)$ and consider $G' = G - xy$



At least one color is missing at both, x, y. We can assume the missing colors are different.

Again the alternating path idea is important: Consider the longest 1-2 alternating path starting at x. The path can not reach y, as bipartiteness holds for the alternating path and P_{12} cannot cross itself. Now swap colors of P_{12} then we can connect $x\,y$ by an edge with color 2. $\square$

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Extremal Graph Theory

Some introductory examples:

E.g.: What is the maximum # edges in an n-vertex graph G that does not contain F as a subgraph.

E.g.: If G does not contain any cycles (i.e. G is a forest

$$|E(G)| \leq n-1 = O(n)$$

E.g.: If G does not contain a K_r topological minor then $|E(G)| \leq c_r \cdot n = O(r)$.

E.g.: If G does not contain K_5 or $K_{3,3}$ as a top-minor, i.e. G is planar, then

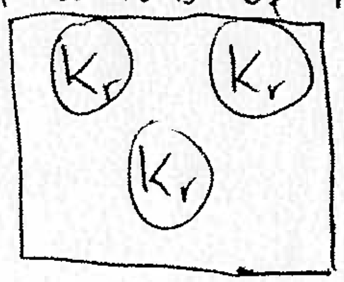
$$|E(G)| \leq 3n-6 = O(n).$$

Thm.: If G does not contain a path of length r, then

$$|E(G)| \leq \frac{r-1}{2} \cdot n$$

where $|G| = n$ and equality holds iff G consists of disjoint unions of K_r.

G =



Path of length r: (Path having r edges)

$$x_0 - x_1 - \ldots - x_r$$

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Lemma: If G is connected on n vertices

degree! $\rightarrow d(x) + d(y) \geq r \quad \forall \; x, y$ nonadjacent vertices.

i) If $r = n$ then G contains a cycle of length n. (such a cycle is said to be Hamiltonian)

ii) If $r < n$ then G contains a path of length r.

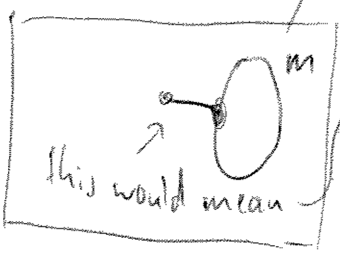
Proof (lemma): Assume the longest path has length $m-1$, i.e.

$$x_1 - x_2 - \dots - x_m .$$

Case 1: G contains a cycle of length m.

Then G contains a Hamiltonian cycle $(m = n)$.

This can be seen by contradiction:

$G =$ 

the loop was not the longest path.

Case 2: G does not contain a cycle of length m.

$$P = x_1 - x_2 - \dots - x_{m-1} - x_m$$

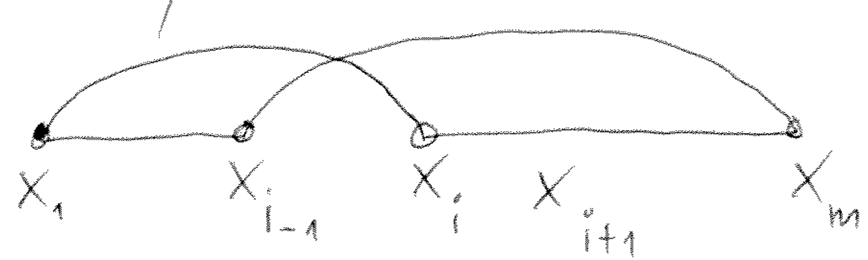
$$r \leq d(x_1) + d(x_m) \; ; \quad x_i \in P,$$

$$N(x_1) = \{ x_i \mid x_1 x_i \in E(G) \}$$

$$N^+(x_m) = \{ x_i \mid x_m x_{i-1} \in E(G) \}$$

"+" because you always pick one vertex to the right.

Note that $N(x_1) \cap N^+(x_m) = \emptyset$, which can be seen by contradiction:



All neighbors must be on path P else maximality of P would be contradicted.

There is no cycle of length m by assumption!
Further

$$r \leq d(x_1) + d(x_m) = |N(x_1)| + |N^+(x_m)| < m$$

$\Rightarrow$ P has length r. $\square$

Proof (thm.): Proof by induction on n. Base case: $n \leq r$,

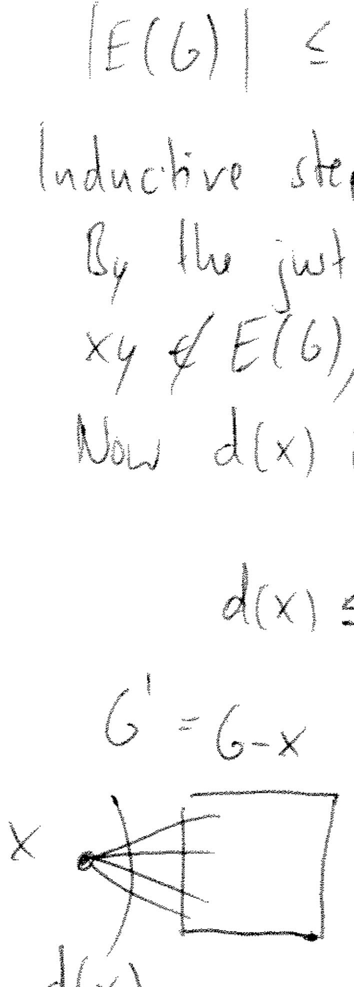
$$|E(G)| \leq \frac{r-1}{2} n \leq \frac{r-1}{2} r = \binom{r}{2}, \text{ always true } \checkmark.$$

Inductive step: $n > r$.

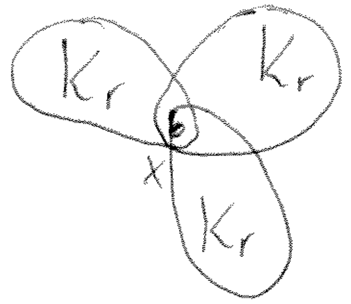
By the just proved lemma, $\exists\ x, y \in V(G)$, $xy \notin E(G)$, s.t. $d(x) + d(y) < r \Rightarrow d(x) < \frac{r}{2}$.

Now $d(x)$ is an integer

$$d(x) \leq \begin{cases} \frac{r}{2} - 1 & \text{if } r \text{ is even} \\ \frac{r}{2} - \frac{1}{2} & \text{if } r \text{ is odd.} \end{cases}$$



$|E(G')| \leq \frac{r-1}{2}(n-1)$, by the inductive hypothesis. Actually $|E(G')| < \frac{r-1}{2}(n-1)$, since we know that G' cannot be the disjoint union of copies of K_r, else G would be



Path of length r would be possible ↯

So $|E(G)| \leq d(x) + \frac{r-1}{2}(n-1)$

$$\leq \frac{r-1}{2} + \frac{r-1}{2}(n-1) = \frac{r-1}{2}n.$$

□

highly exam relevant
↓

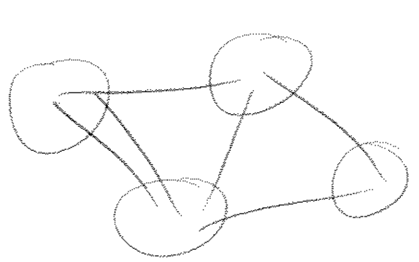
Thm.: (Turán's theorem)

If G does not contain a clique of size $r+1$, i.e. no K_{r+1} as a subgraph, then

$$|E(G)| \leq \left(1 - \frac{1}{r}\right)\frac{n^2}{2}$$

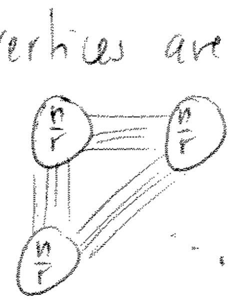
Def.: G is said to be r-partite if $V(G) = \dot{\bigcup}_{i=1}^{r} V_i$ with $uv \in E(G) \Rightarrow u \in V_i$ and $v \in V_j$ for $i \neq j$.

E.g.



No edge inside any of the r parts.

Def.: The Turan graph on n vertices, r-partite, $T_{n,r}$, is the complete r partite graph s.t. each part has $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$ vertices. So the vertices are distributed as evenly as possible: $T_{n,r} =$

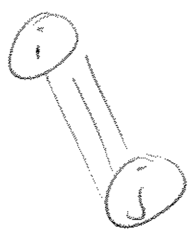


The difference between # vertices between any two parts is 1

$$E(T_{n,r}) = \frac{n^2}{r^2}\binom{r}{2} = \left(1-\frac{1}{r}\right)\frac{n^2}{2}$$

Lemma: Among all r-partite graphs, $T_{n,r}$ has the greatest # edges.

Proof: By contradiction: Assume $\|G\| = |E(G)|$ is maximal and $i-j \geq 2$

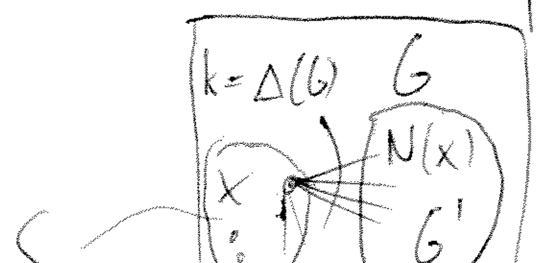
$G =$ 

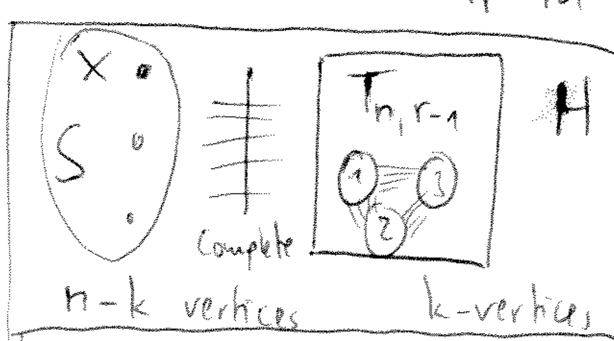
(i,j vertices in these parts, respectively.)

Then ~~woul~~ one could gain an edge by reconnecting G, so it was not maximal:

$i-1-j = i-j-1 \geq 2-1 = 1$, one edge gained! $\square$

Proof (Turán): Proof by induction on r. The base case, $r=1$ holds. For the inductive step, $r>1$, Construct a r-partite graph H s.t. $|E(G)| \leq |E(H)| \leq |E(T_{n,r})| \leq \left(1-\frac{1}{r}\right)\frac{n^2}{2}$, which will be our plan. There is no K_r in G'.





H is an r-partite graph and $|E(G')| \le |E(H')|$ where $H' = T_{n,r-1}$.

$$|E(G)| \le |E(G')| + \sum_{x \in S} d(x) \overset{\text{ind. hyp.}}{\le} |E(H')| + \sum_{x \in S} d(x)$$

$$\overset{\text{since } \Delta(G)=k}{\le} |E(H')| + k(n-k) \le |E(H)| \overset{\text{lemma}}{\le} |E(T_{n,r})| \overset{\text{by def. of } T_{n,r}}{\le} \left(1-\frac{1}{r}\right)\frac{n^2}{2}$$

$\square$

Next let us consider the bipartite version of Turán's theorem.

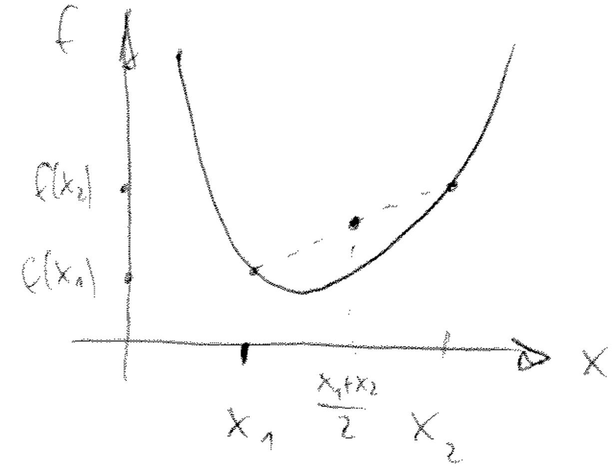
Thm.: Let G be bipartite s.t. $V(G) = V_1 \cup V_2$ s.t. $|V_1| = m$ and $|V_2| = n$ where $K_{r,s}$ is not a subgraph of G, then

$$|E(G)| \le C_{r,s} \cdot n \cdot m^{1-1/s} + \frac{sm}{2} \quad \text{or}$$

$$C_{r,s} \cdot m \cdot n^{1-1/r} + \frac{rn}{2}$$

Proof: We'll use Jensen's inequality: If $f: \mathbb{R} \to \mathbb{R}$ is concave up, then $\frac{f(x_1)+\ldots+f(x_n)}{n} \ge f\left(\frac{x_1+\ldots+x_n}{n}\right)$

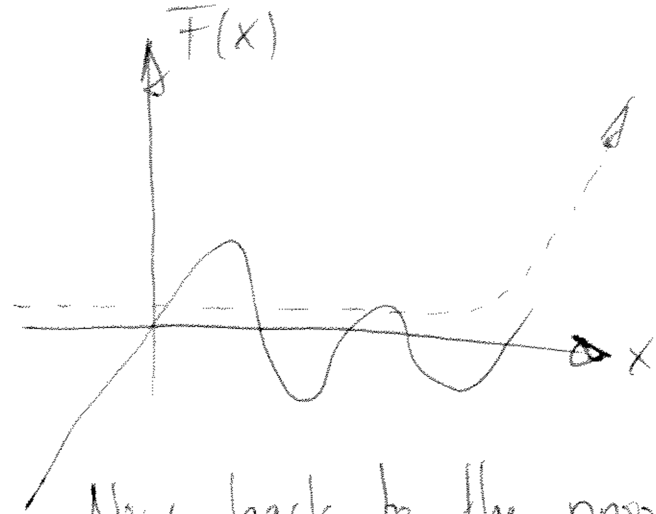
as the following sketch illustrates:



$$\frac{f(x_1)+f(x_2)}{2} \geq f\left(\frac{x_1+x_2}{2}\right)$$

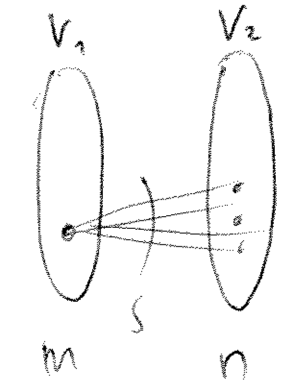
We want to apply Jensen's inequality to $F(x) := \binom{x}{s}$

$$= \begin{cases} \frac{x(x-1)(x-2)\dots(x-(s-1))}{s!} & \text{if } x \geq s \\ 0 & \text{else} \end{cases}$$



$\Rightarrow$ a concave up function.

Now back to the proof of the theorem.



By double counting of # "stars", i.e. vertices in m that are connected to s vertices in n. Consider this big overestimate:

$\#\text{stars} \leq \binom{n}{s}\cdot(r-1)$, since we have no $K_{r,s}$ by assumption. Clearly, $\frac{(n-s)^s}{s!} < \binom{n}{s} < \frac{n^s}{s!}$

$$\sum_{x\in V_1}\binom{d(x)}{s} = \#\text{stars}.$$

Applying Jensen's inequality:

$$m\binom{\frac{E(G)}{m}}{s} = m\cdot\binom{\frac{\sum_{x\in V_1} d(x)}{m}}{s} \overset{\text{Jensen}}{\leq} \sum_{x\in V_1}\binom{d(x)}{s} \leq \binom{n}{s}(r-1)$$

$$m\frac{\left(\frac{|E(G)|}{m}-s\right)^s}{s!} \leq m\binom{\frac{E(G)}{m}}{s} \leq \frac{n^s}{\cancel{s!}}(r-1)$$

$$\left(\frac{|E(G)|}{m}-s\right)^s \leq n^s m^{-1}(r-1)$$

$$\frac{|E(G)|}{m} \leq n\cdot m^{-1/s}(r-1)^{1/s}+s$$

$$|E(G)| \leq n\cdot m^{1-\frac{1}{s}}(r-1)^{\frac{1}{s}}+sm \qquad \square$$

Extremal Graph Theory

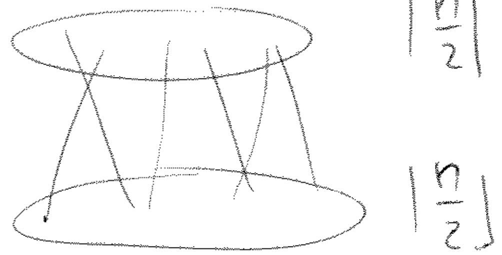
A special case of Turán's theorem:

Thm.: (Mantel's thm.)

Every triangular-free graph with n vertices has at most $\frac{n^2}{4}$ edges. I.e. $|E(G)| \leq \frac{n^2}{4}$.

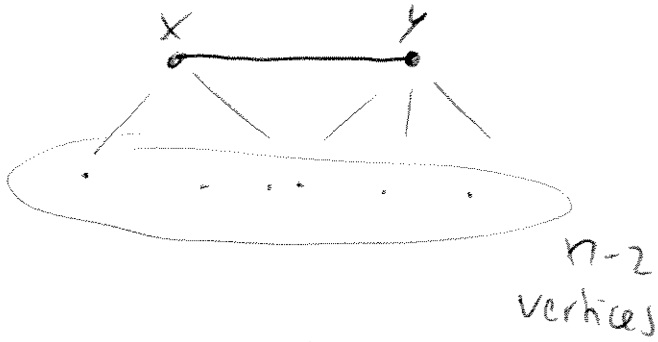
Triangle-free $\Rightarrow \omega(G) = 2$.

Proof:



This graph will have $\lfloor \frac{n^2}{4} \rfloor$ edges. (It's bipartite and completely connected).

① Consider an edge:



$(d(x)-1) + (d(y)-1) \leq n-2$,
since G is triangle-free,

$d(x) + d(y) \leq n$, for any x, y, s.t. $xy \in E(G)$.

Summing up this inequality for every $E(G)$ gives

$$\sum_{xy \in E(G)} \left(d(x) + d(y) \right) \leq |E(G)| \cdot n$$

(since $xy = yx$)
$$\sum_{x \in V(G)} d^2(x) \leq |E(G)| \cdot n.$$

Recall Jensen's inequality, a simple consequence of convexity.

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$$F(x) = \binom{x}{s}$$

Let $f(x)$ be convex, e.g. $f(x) = x^2$, then

$$\frac{\sum_{\text{all } x}^{n} f(x)}{n} \geq f\left(\frac{\sum^{n} x_i}{n}\right).$$

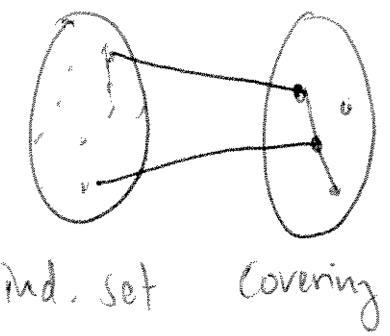
Applied to our inequality:

$$\frac{1}{n}\left(\sum_{x \in V(G)} d(x)\right)^2 \leq \sum_{x \in V(G)} d^2(x) \leq |E(G)|\, n$$

$$\Rightarrow \quad \left(2|E(G)|\right)^2 \leq |E(G)|\, n^2 \quad \Rightarrow \quad |E(G)| \leq \frac{n^2}{4}. \ \square$$

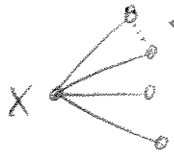
(2) Alternative proof:

Let G be K_3-free and recall $\alpha(G) :=$ maximal # independent vertices of G, $\tau(G) :=$ covering number of G = min # vertices such that every edge of G contains at least one of them.



The complement of an independent set is a covering set. Hence $\alpha(G) + \tau(G) = n$.

Since G is K_3-free, $d(x) \leq \alpha(G)$, since



$N(x)$ must be independent.

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Let S be a minimal covering set.

$$|E(G)| \leq |S| \cdot \alpha(G) = \tau(G) \cdot \alpha(G) \leq \left(\frac{\tau(G)+\alpha(G)}{2}\right)^2$$

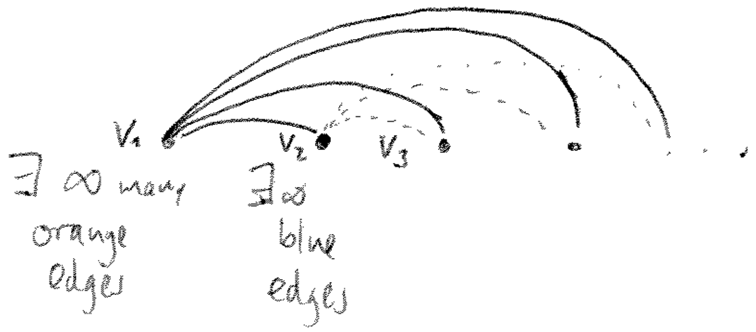
$$= \frac{n^2}{4}, \quad \text{since } \forall a,b \in \mathbb{R}, \quad ab \leq \left(\frac{a+b}{2}\right)^2. \square$$

Turán's theorem can be read at a theorem from mathematical logic; If there are sufficiently many relations in a group then it will be easier to find mutually related ones.

Any two vertices are either connected by an orange or blue edge. Blue means the edge is not there.

Thm.: (Ramsey's)
Let G be an infinite graph, 2-colored complete.
There are always infinitely many vertices such that any 2 are connected by an orange edge, or infinitely many s.t. any 2 are connected by an blue edge.
(Infinite Ramsey theorem)

Proof: As often, this proof starts by a construction. Suppose there are infinitely many orange ~~vertices~~ edges adjacent to the first vertex.



Now only consider the infinitely many vertices which are connected to v_1 via orange and to v_2 via blue edge and to v_3 via blue edge.

graph 09/05 4

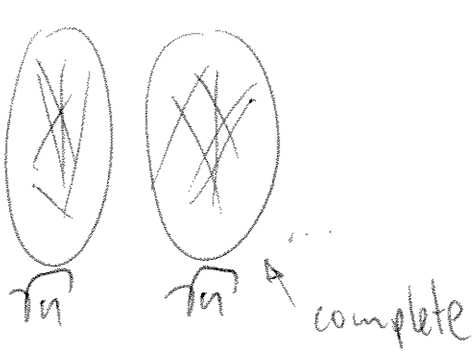
This is an infinite sequence of vertices with the following property: $\forall v_i$ either all edges that go from v_i to higher labelled vertices are orange or all blue.
Suppose that for infinitely many vertices they are only connected via orange edges to vertices with greater label. But this means we have found infinitely many ~~oran~~ vertices that are all completly connected with orange edges. $\square$

clique 

Thm.: (Finite Ramsey thm.)

Every sufficiently large complete graph whose edges are either orange or blue, contains many vertices ($\geq r$) such that any two are connected by an orange edge (or blue for all).
I.e. every graph with $|V(G)| \geq R(r)$ satisfies that
$\omega(G) \geq r$ or $\alpha(G) \geq r$.

Turán's conjecture:



$R(r) \leq r^2$

FALSE!

Proof: A little asymmetry helps here. We wand $R(r)$ but start:
Let $R(r,s)$ = min # vertices s.t. one can always find r vertices completely connected in one color ~~and~~ or s vertices completely connected in the other color.
$R(1,s) = 1 = R(r,1)$.

graph 09/05 5

$R(2,s) = s$, $R(r,2) = r$

Lemma: (Erdös-Szekeres)

$$R(r,s) \leq R(r-1,s) + R(r,s-1)$$

Proof: Just like in the infinite case: $n = R(r-1,s) + R(r,s-1)$

orange →  $\geq R(r-1,s)$ ← suppose w.l.o.g.

blue → $\geq R(r,s-1)$. $\square$

Applying the lemma, we obtain by induction that

$R(s,s) \leq \binom{r+s-1}{r-1}$ $\forall r,s$, since the lemma

says $R(r,s) \leq \binom{r-1+s-1}{r-2} + \binom{r+s-2}{r-1} = \binom{r+s-1}{r-1}$

induction hypothesis.

let $R(r,r) := R(r)$, then

$R(r) \leq \binom{2r-1}{r-1} < 2^{2r} = 4^r$. $\square$

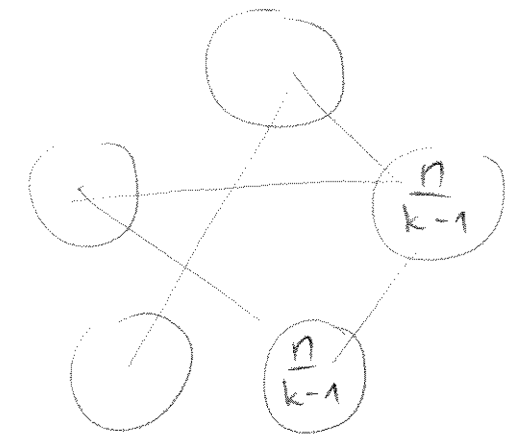
graph 16/05 1

Thm. (Turán)

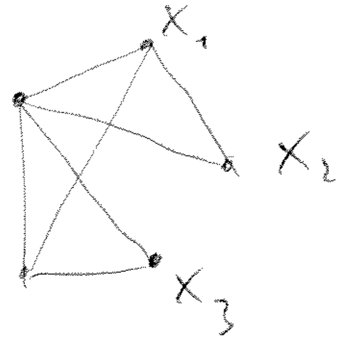
If $K_k \not\subseteq G$, $k \geq 2$, then with $|V(G)| = n$,

$$|E(G)| \leq \frac{n^2}{2}\left(1 - \frac{1}{k-1}\right).$$

If $k-1 \mid n$ then



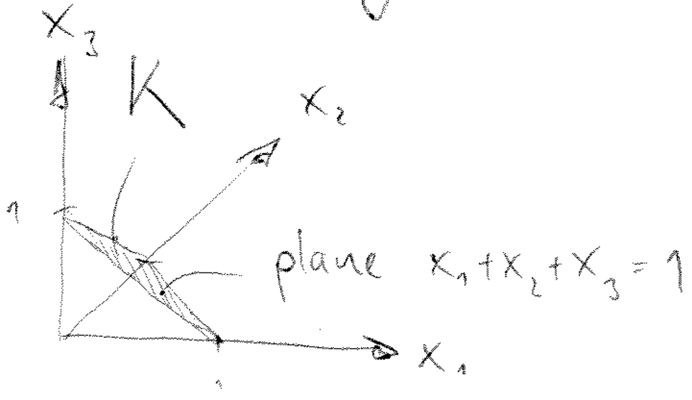
Proof: (Analytical proof by Motzkin and Straus)



Assign a variable x_i to each of the n vertices of the graph G.

$$f(x_1, x_2, \ldots, x_n) := 2 \sum_{i,j \in E(G)} x_i x_j \quad \text{and let}$$

Region $K := \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1, x_2, \ldots, x_n \geq 0$ and $\sum_{i=1}^{n} x_i = 1 \}$, a compact region, i.e. closed and bounded. E.g. with $n = 3$:



A continuous function defined on a compact region must attain its maximum on that region, i.e. here K.

max of f on K

let $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ be such a point $\bar{p}$ s.t. the # strictly positive coordinates is minimal. Maybe there are more such points, but there is one where the # zero coordinates is maximal.

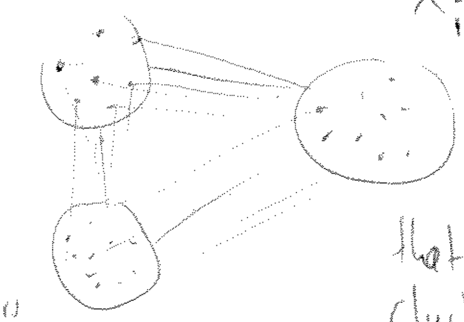
Claim: $C = \{v_i \in V(G) \mid \bar{x}_i > 0\}$ is a clique in G.
Assume w.l.o.g. $\bar{x}_1 > 0$, $\bar{x}_2 > 0$ but $12 \notin E(G)$ for the sake of contradiction.
Consider $\bar{p} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n)$ and shift it now a bit. Say $\bar{x}_1 + t$, $\bar{x}_2 - t$, since we leave $\bar{x}_i$ fixed $\forall\, i \notin \{1,2\}$ and $\sum_{\text{all } i} x_i = 1$. So t can be seen as time. If $t \in [-\bar{x}_1, \bar{x}_2]$ then $\bar{x}_1 + t$ and $\bar{x}_2 - t$ are non-negative.
So $p_t = (\bar{x}_1 + t, \bar{x}_2 - t, \bar{x}_3, \ldots, \bar{x}_n) \in K$. Next consider $f(p_t)$ and note that we assumed $12 \notin E(G)$, thus $f(p_t)$ will change linearly with t.
At one of the endpoints of the interval $[-\bar{x}_1, \bar{x}_2]$, p_t will be better than p, because one more $\bar{x}_i = 0$ for it.
This contradiction proves the claim.
Consider $f(\bar{p} = (\bar{x}_1, \ldots, \bar{x}_n))$ and the condition $\sum_i x_i = 1$,
so $1 = (\bar{x}_1 + \bar{x}_2 + \ldots + \bar{x}_n)^2 = \left(\sum_{i \in C} \bar{x}_i\right)^2 = \underbrace{2 \sum_{i \neq j \in C} \bar{x}_i \bar{x}_j}_{f(\bar{x}_1, \ldots, \bar{x}_n)} + \sum_{i \in C} \bar{x}_i^2$.
But $i, j \in C \iff ij \in E(G)$.

graph 16/05 3

But Jensen's inequality holds and so $\sum_{i\in C} \bar{x}_i^2$ is minimized if all $\bar{x}_i$ are the same:

$$\frac{\sum_{i\in C} \bar{x}_i^2}{|C|} \geq \left(\frac{\sum_{i\in C} \bar{x}_i}{|C|}\right)^2$$

$f \propto m$



So positive weights only on the clusters that form a clique

X_i = percentage of all vertices n that are in cluster i.

$$f(\bar{x}_1, \dots, \bar{x}_n) \leq 2\sum_{i,j\in C} \bar{x}_i\bar{x}_j \leq 2\cdot\binom{|C|}{2}\cdot\frac{1}{|C|^2} = 1 - \frac{1}{|C|}$$

$$\max_K f \leq 1 - \frac{1}{k-1}$$

let $p' := \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right) \in K$, then

$$f(p') = 2|E(G)|\cdot\frac{1}{n^2} \leq \max_K f \leq 1 - \frac{1}{k-1}.$$

$$m \leq \frac{n^2}{2}\left(1 - \frac{1}{k-1}\right). \quad \square$$

graph 16/05 4

Thm.: (Ramsey)

$\forall r\ \exists\ R(r)$ s.t. $\forall G$ with $n \geq R(r)$ contains either a complete graph K_r with r vertices or an independent set of r vertices.

Lemma: $R(r,s) \leq R(r-1,s) + R(r,s-1)$

Thm.: $R(r) \leq \binom{2r-1}{r-1} < 4^r$.

} last time

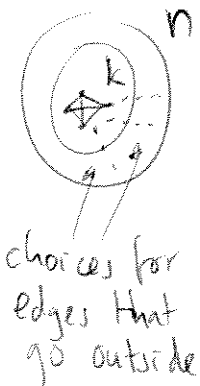
(r−1) replace

Thm.: (Erdős) $2^{r/2} \leq R(r) \iff \exists$ a graph with $2^{r/2}$ vertices that has no K_r and no ind. set of size r.

Proof: A graph G is "bad" if $G \supseteq K_r$ or G has r ind. vertices. We would like to show that # bad graphs on n vertices < total # graphs on n vertices $= 2^{\binom{n}{2}}$. So what is # bad graphs?

Let K be a fixed r-element subset of $\{1,2,\dots,n\}$.

If K induces a complete graph on G, the possible # of such "bad" graphs is $2^{\binom{n}{2}-\binom{k}{2}}$



Similarly, if K induces an independent set in G then

# of such "bad" graphs is also $2^{\binom{n}{2}-\binom{k}{2}}$

Hence # bad graphs $= 2 \cdot 2^{\binom{n}{2} - \binom{k}{2}}$, and thus

bad sets $< \binom{n}{r} 2 \cdot 2^{\binom{n}{2} - \binom{k}{2}}$, a very rough

upper bound. To show:

$$\binom{n}{r} 2 \cdot 2^{\binom{n}{2} - \binom{k}{2}} < 2^{\binom{n}{2}}$$

$$\frac{2 \cdot \binom{n}{r}}{2^{\binom{k}{2}}} < 1 \Leftrightarrow 2\binom{n}{r} < 2^{\binom{k}{2}}$$

$$\binom{n}{r} < \frac{n^r}{r} \Rightarrow \quad 2n^r < r\, 2^{\binom{r}{2}}$$

$$n < \underbrace{\left(\frac{r}{2}\right)^{1/r}}_{>1} 2^{\frac{r-1}{2}}$$

$$n < 2^{\frac{r-1}{2}} \checkmark \quad \square$$

More precisely: $R(r) \geq r \cdot \sqrt{2}^r$

This proof can also be done via the probabilistic method.

graph 23/05 1

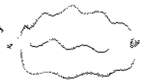
List of theorems must be known with proofs. One proof on blackb
Put together a list of theorems proved in class and send it via e-mail to Janos. pachjanos@gmail.com

Exam: 2 problems to be solved and one theorem proved. 30 min to solve problems, 30 minutes to present solutions and the theorem.

Flows in Networks (Ford-Fulkerson)

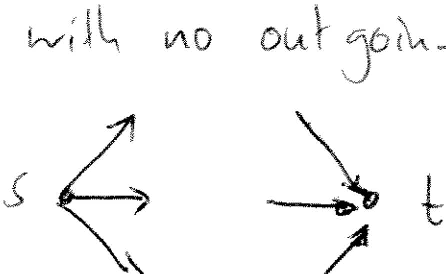
This will be another example of min-max theorems of which we have encountered already some. E.g.

König's thm. : max matching = min cover

Menger's thm. : a  b

Those are corollaries of Ford-Fulkerson, so this is the mother of min-max theorem. One can argue that linear programming is behind Ford-Fulkerson.

Def.: A Network is a directed graph G, i.e. each edge has an orientation, $e = \overrightarrow{uv}$. Each network has a sink, $t \in V$ and a source, $s \in V$. A source is a vertex with no incoming edges and a sink is a vertex with no outgoing edges. (They are unique!)

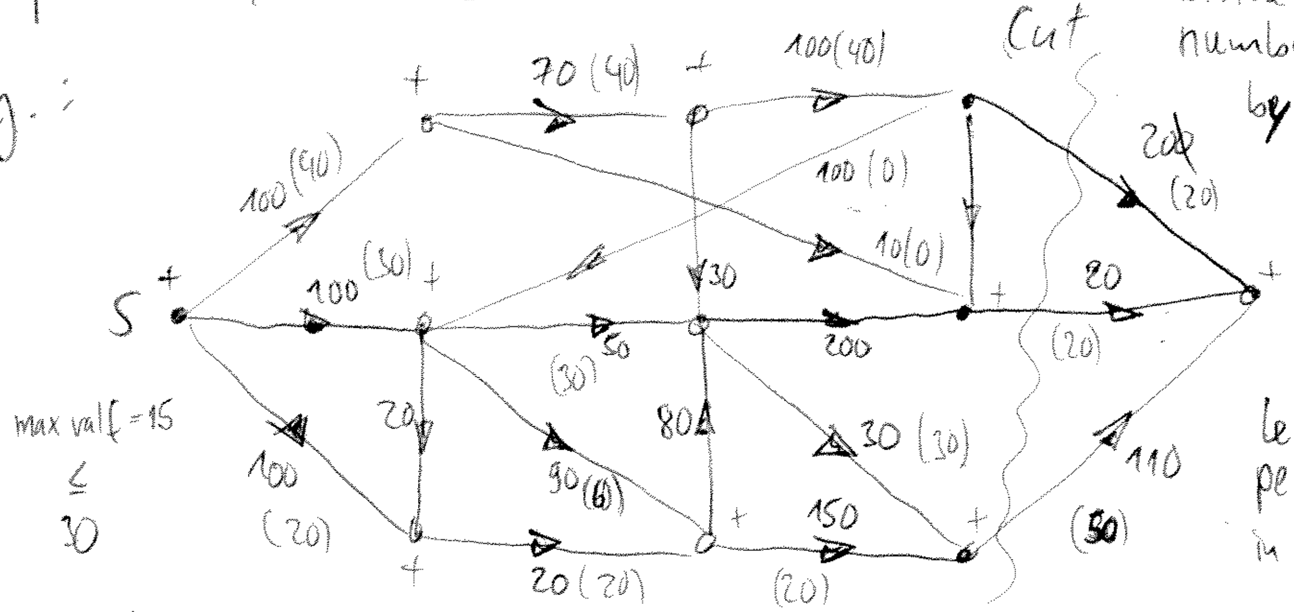


Every edge in a network has a capacity $c(\overrightarrow{uv}) > 0$

graph 23/05 2

E.g.:

Divide each number (c) by 10.



Let the flows per edge be in bracket

The capacity can be seen as a flow rate when interpreting the network as a water supply network.

Def.: $f : \begin{cases} E(G) \longrightarrow \mathbb{R}^+ \\ \vec{uv} \longmapsto f(\vec{uv}) \end{cases}$ s.t. $0 \le f(\vec{uv}) \le c(\vec{uv})$.

$f :=$ flow

Kirchoff's law says that flow can appear or disapper in the source or the sink. I.e.

$$\sum_{\vec{uv} \in E} f(\vec{uv}) = \sum_{\vec{vz} \in E} f(\vec{vz}) \quad \forall \; v \in V(G), \text{ i.e.}$$

incoming flow = outgoing flow (conservation of charge/water e.g.)

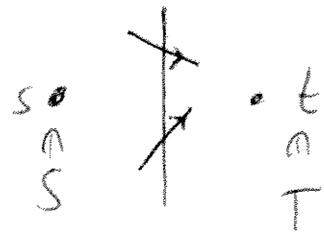
E.g.: The least capacity of an edge limits the flow through a path, a bottle neck.

Lemma 1: $$\sum_{\vec{sv} \in E} f(\vec{sv}) = \sum_{\vec{vt} \in E} f(\vec{vt}) := \operatorname{val} f.$$ (Same flow in sink/source)

Problem: $\max\limits_{f} \operatorname{val} f = ?$

graph 23/05 3

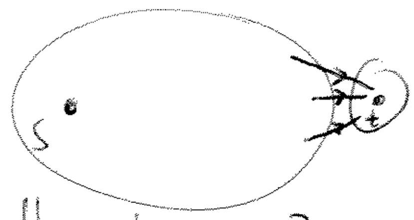
Lemma 2: Partition the vertices of G into S and T s.t. $s \in S$ and $t \in T$.



Let $\mathcal{F}(S,T) = \sum_{u \in S, v \in T} f(\vec{uv})$, i.e. the total flow from S to T, $- \sum_{u \in T, v \in T} f(\vec{uv})$, i.e. flow from T to S. So the

claim is: $\mathcal{F}(S,T) = \sum_{u \in S, v \in T} f(\vec{uv}) - \sum_{u \in T, v \in S} f(\vec{uv}) = \underbrace{\sum_{\vec{su} \in E} f(\vec{su})}_{\text{total flow out of } s}$.

Proof of lemma 1: Consider the following partition:



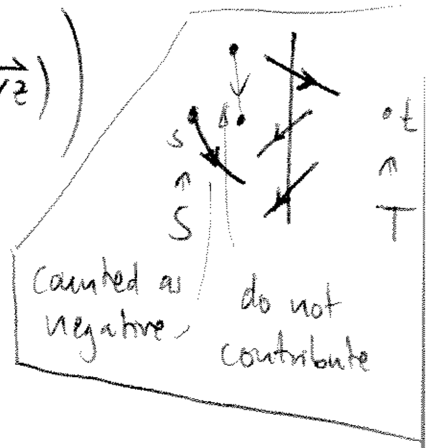
$T = \{t\}$, $S = V - \{t\}$.

Here we have no backflow, thus lemma 2 says: $\mathcal{F}(S,T) = \sum_{\vec{vt} \in E} f(\vec{vt}) = \sum_{\vec{su} \in E} f(\vec{su})$. $\square$

Proof lemma 2: Add up Kirchoff's law for all $v \in S \setminus \{s\}$:

$$0 = \sum_{v \in S \setminus \{s\}} \left(\sum_{\vec{uv} \in E} f(\vec{uv}) - \sum_{\vec{vz} \in E} f(\vec{vz}) \right)$$

$$= \sum_{\vec{sv} \in E} f(\vec{sv}) - \sum_{\substack{u \in S \\ v \in T}} f(\vec{uv}) + \sum_{\substack{u \in T \\ v \in S}} f(\vec{uv})$$



$$= \sum_{\vec{sv} \in E} f(\vec{sv}) - \mathcal{F}(S,T). \quad \square$$

graph 23/05 4

So lemma 2 can be reformulated:

capacity of the partition (cut) $:= c(S,T)$

Lemma 2:

$$\text{val} f = \sum_{\substack{u\in S\\ v\in T}} f(\vec{uv}) - \sum_{\substack{u\in T\\ v\in S}} f(\vec{uv}) \le \underbrace{\sum_{\substack{\vec{uv} \text{ s.t.}\\ u\in S\\ v\in T}} c(\vec{uv})}$$

for any flow f and any partition $S \cup T = V$.

Thm.: (Ford-Fulkerson) "Max-flow-Min-Cut Theorem"

$$\max_{f,\text{ a flow}} \text{val} f = \min_{\substack{\text{all partitions}\\ S\cup T = V}} c(S,T)$$

Proof: $\max_f \text{val} f \le \min_{S\cup T=V} c(S,T)$ by lemma 2. The other inequality will be proved via the following marking algorithm:

marked by $\oplus$

Rule A: If $u^+ \xrightarrow{f<c} v$ then mark v by a $\oplus$.

Rule B: If $u^+ \xleftarrow{f>0} v$ then mark v by a $+$.

Start the algorithm by marking the source: s^+.

Claim I: If at the end of the procedure t is marked, then the value of f is not maximal, i.e. it can be improved.

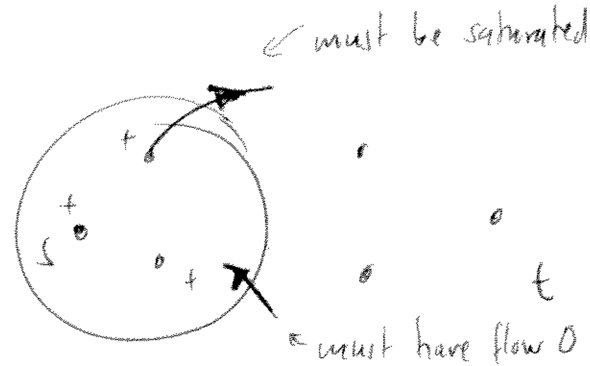
For, $s^{*} \xrightarrow{6(4)} \bullet^{+} \longleftarrow \bullet^{+} \longrightarrow t^{+}$, so one can send more water along this path. At least one extra flow unit can be send.

by rule B

graph 23/05 5

Claim II: If t is not marked by the algorithm, then the flow is optimal.

For,



Let the set of marked vertices be S and the set of unmarked vertices be T.

If there is an outgoing edge from S to T it must be saturated, by rule A. An ingoing edge from T to S must be used at f = 0 by rule B. But then, with this special partition, lemma 2 assures $\mathrm{val} f = \sum_{\substack{u \in S \\ v \in T}} c(\vec{uv}) = c(S,T)$ so the optimal flow value.

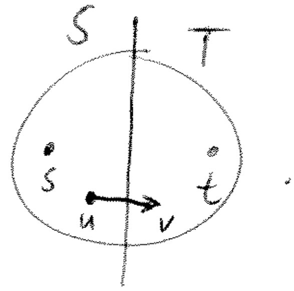
So we found a partition S, T where $\max_{f \text{ flow}} \mathrm{val} f = \min_{\substack{\text{partitions} \\ \text{cuts } S,T}} c(S,T)$.

Recall: Ford-Fulkerson Thm. we discussed last time.

Thm.: (Ford-Fulkerson)

$$\max_{\text{flows } f} \operatorname{val} f = \min_{\substack{\text{cut } (S,T) \\ V = S \cup T,\ s \in S,\ t \in T}} c(S,T)$$

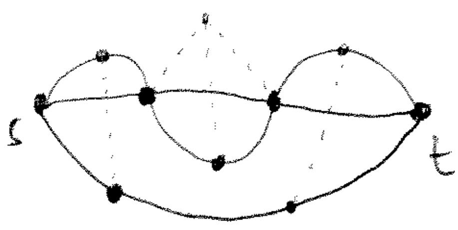
where $c(S,T) = \sum_{\substack{u \in S \\ v \in T}} c(\overrightarrow{uv})$



The algorithm to proof this theorem last time has the immediate consequence that if the capacities have integer values also the resulting flow must have integer values.

Corollary: (Menger's thm, undirected edge version)

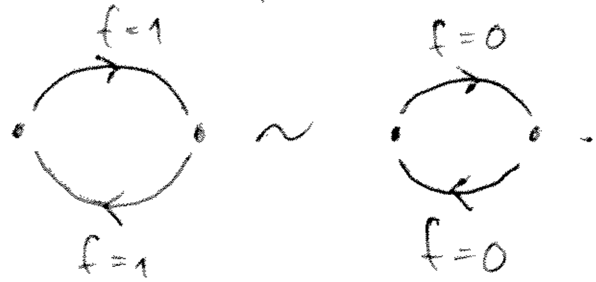
(max # edge-disjoint paths between s and t)
= (min # edges whose removal disconnects s and t)



Note: One inequality is often trivial in min-max theorems.

graph 30/05 2

Proof: Apply FFT for undirected graphs. Now $c(uv) = c(vu) = 1$. $\exists$ maximal flow for which $f(\vec{uv}) \in \{0, 1\}$ as:

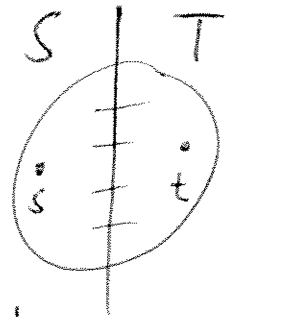


Note that FFT has a different cut definition for undirected graphs: $c(S,T) = \sum_{u \in S, v \in T} c(u,v)$.

Clearly $f(\vec{uv}) \le c(uv)$.

By FFT:

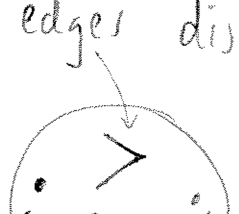
$$\max_f \operatorname{val} f = \min_{S \cup T = V} c(S,T)$$



Now $\min_{S \cup T = V} c(S,T) = \min_{S \cup T = V}$ # edges between **S** and **T**

= min # edges whose removal disconnects s and t,

since the removal of those edges disconnects s and t, let S consist of the points that can still be reached from s, and $T := V \setminus S$.



To complete the proof it remains to show that

$$\max \operatorname{val} f = \max \text{ \# edge disjoint paths } s\text{-}t$$

but this f is obvious. $\square$

graph 30/05 3

Corollary: (König's Thm.)

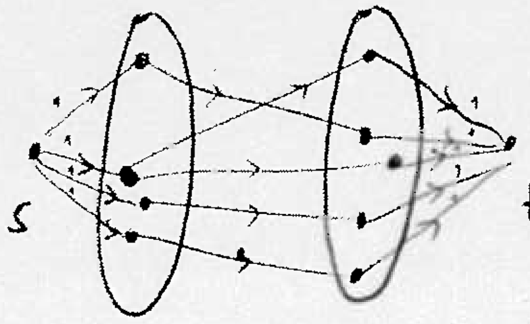
In a bipartite graph, $V = A \cup B$,

(max size of a matching) = (min # vertices to cover all edges)

$\leq$

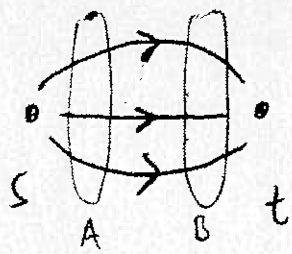
$\uparrow$ is trivial

Proof:



Introduce a sink t and a source s, connect all vertices from A with s and all from B to t as sketched. Now we can apply directed FFT.

We also have to introduce capacities. Equip each edge connecting A to B with capacity $|V|+1$ (this is like ∞) and all edges from s to A and B to t with capacity 1. Since $f(\vec{uv}) \in \{0,1\}$, each edge is either used or not in a maximal flow.



So $\max_f \operatorname{val} f$ = (max size of a matching).

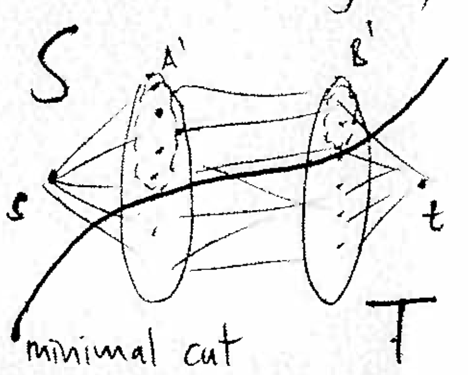
Can we also say that

$\min c(S,T)$ = (min # vertices that cover all edges) ?

$\leq$ ← trivial

$\geq$?

Consider the minimum cut, maybe



graph 30/05 4.

Let $A' = S \cap A$ and $B' = S \cap B$. There are no edges from A' to $B \setminus B'$ by optimality of the flow. As there are no edges from A' to $B \cap T$,

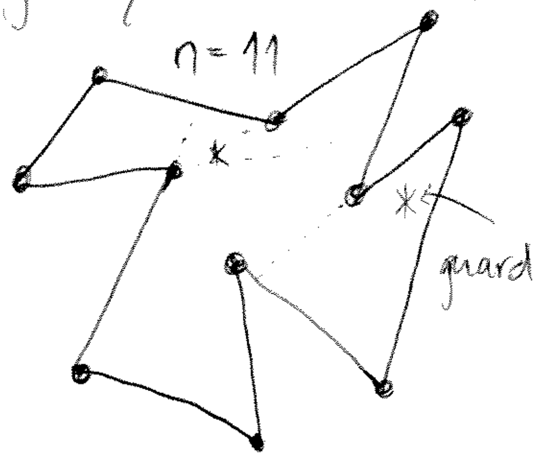
$(A \setminus A') \cup B'$ covers all edges of our bipartite graph. So

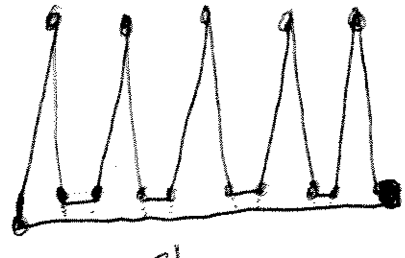
$c(S,T) = |A \setminus A'| + |B'| \geq$ (min # vertices to cov. all e

which completes the proof. $\square$

Thm.: (Chvatal's Thm. = Art Gallery Thm.)

Consider a closed polygon ("art gallery") in which we want to place the smallest number of guards necessary to monitor every point in the gallery. For example for this 11-gon, two guards suffice but 1 is not enough.





For the second example k guards are necessary and sufficient. The theorem says now that

$\left\lfloor \frac{n}{3} \right\rfloor$ guards are sufficient to monitor any art gallery with n vertices.

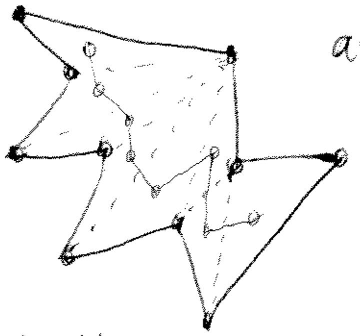
graph 30/05 5

Proof: (By Fisk)

non adjacent

i) Triangulate the art gallery (possible, since any polygon has 2 vertices that can be connected by a segment within the polygon. The generalisation can be done via induction). Why?

ii) The resulting graph of n vertices has 2 vertices of degree 2. This can be seen in the following way. Place a vertex ($\circ$) in each triangle and connect two vertices if the corresponding triangles share an edge. This produces a tree, since the connectedness of the path is obvious and the acyclicity follows by contradiction:



If there was a cyclic path then it would have to cross the art gallery boundary, which is impossible. But every tree has at least two leaves which guarantees the existence of 2 vertices of degree 2 in the triangulated art gallery.

iii) The triangulated graph G is three-colorable, i.e. $\chi(G) \leq 3$, (that can be proved by induction).

iv) Take the smallest color class $\leq \left\lfloor \frac{n}{3} \right\rfloor$, by pigeon-holing. It's vertices monitor the gallery. $\square$

(if all triangular cells are monitored, the whole gallery is monitored.)