

# Combinatorial Optimization – Problem Set 9 – Solutions

You can hand in one of the following problems at the start of Tuesday's problem session. Please explain your solution carefully. Don't forget to put your name.

## NP-Hardness

1. Show that the following two optimization problems are  $\mathcal{NP}$ -hard:

**INDEPENDENT SET:** Given an undirected graph  $G$ , find a maximum cardinality independent set, i.e. a set  $I \subset V(G)$  such that  $E(I) = \{uv \in E(G) : u, v \in I\} = \emptyset$ .

**CLIQUE:** Given an undirected graph  $G$ , find a maximum cardinality clique, i.e. a  $K \subset V(G)$  such that  $uv \in E(G)$  for all  $u, v \in K$ .

We first reduce VERTEX COVER to INDEPENDENT SET, which proves that INDEPENDENT SET is  $\mathcal{NP}$ -hard. Given  $G$  in which to find a minimum vertex cover, consider  $G$  as an instance of INDEPENDENT SET.

If  $I$  is an independent set, then  $C = V(G) \setminus I$  is a vertex cover. Indeed, if  $e \in E(G)$ , then one of its endpoints is not in  $I$ , so it is in  $C$ , hence it covers  $e$ . And  $|C| = |V(G)| - |I|$ , so  $C$  is minimum if and only if  $I$  is maximum.

Next we reduce INDEPENDENT SET to CLIQUE. Given  $G$  in which to find a maximum independent set, we consider its complement  $\overline{G}$  as an instance of CLIQUE. Recall that  $V(\overline{G}) = V(G)$  and

$$E(\overline{G}) = \{uv : u, v \in V(G), uv \notin E(G)\}.$$

Now if  $K \subset V(G)$  is a clique in  $\overline{G}$ , then  $K$  is an independent set in  $G$ . So clearly maximum cliques in  $\overline{G}$  correspond to maximum independent sets in  $G$ .

Note that in both cases, the reduction can easily be reversed, so in fact these 3 problems are what is called *polynomially equivalent*.

2. Show that the following optimization problem is  $\mathcal{NP}$ -hard:

**LONGEST PATH:** Given a directed graph  $G$  with weights  $w : E(G) \rightarrow \mathbb{R}$ , and  $s, t \in V(G)$ , find a directed path from  $s$  to  $t$  of maximum weight.

We will reduce SHORTEST PATH (find a shortest path in a weighted directed graph between two given vertices, allowing negative cycles) to LONGEST PATH. In Problem Set 3, we already saw that HAMILTON CYCLE reduces to SHORTEST PATH, so this implies that LONGEST PATH is  $\mathcal{NP}$ -hard.

Given  $G, w$  for which to find a shortest path, define  $w'$  by  $w'(e) = -w(e)$ , and consider  $G, w'$  as an instance of LONGEST PATH. Clearly, a longest path for  $w'$  corresponds exactly to a shortest path for  $w$ .

3. Show that the following optimization problem is  $\mathcal{NP}$ -hard:

INTEGER PROGRAMMING: Given a matrix  $A \in \mathbb{Z}^{m \times n}$  and vectors  $b \in \mathbb{Z}^m, c \in \mathbb{Z}^m$ , find a vector  $x \in \mathbb{Z}^n$  such that  $Ax \leq b$  and  $cx$  is maximum, if possible.

Probably the easiest way for us to do this is to reduce INDEPENDENT SET to INTEGER PROGRAMMING. Given a graph  $G$  in which to find a maximum independent set, we take the integer program that maximizes  $\sum_{v \in V(G)} x_v$  subject to  $x_v \in \mathbb{Z}, 0 \leq x_v \leq 1$  for all  $v \in V(G)$  and  $x_u + x_v \leq 1$  for all  $uv \in E(G)$ . This works because a vertex set is independent if and only if it contains at most one endpoint of every edge. More precisely, we take  $n = |V(G)|$ ,  $m = |E(G)| + 2|V(G)|$ ,  $c = \bar{1}$ , and

$$A = \begin{bmatrix} M^T \\ I \\ -I \end{bmatrix}, \quad b = \begin{bmatrix} \bar{1} \\ \bar{1} \\ 0 \end{bmatrix},$$

where  $M$  is the incidence matrix of  $G$ .

4. Show that the following optimization problem is  $\mathcal{NP}$ -hard:

METRIC TSP: Let  $G$  be a complete undirected graph  $G$  with a weight function  $d : E(G) \rightarrow \mathbb{R}_{>0}$  that satisfies the triangle inequality

$$d(uw) \leq d(uv) + d(vw)$$

for all  $u, v, w \in V(G)$ .

Find a minimum weight Hamilton cycle in  $G$ .

We reduce HAMILTON CYCLE to METRIC TSP by almost the same construction that we used in the notes, but we give the edges weights 1 and 2 instead of 1 and 10. More precisely, given  $G$  in which to find a Hamilton cycle, give all its edges weight 1, then add edge between any two unconnected vertices, and give these new edges weight 2. Call this new graph  $G'$ .

Then the minimum weight Hamilton cycle in  $G'$  has weight  $|V(G)|$  if and only if it is also a Hamilton Cycle in  $G$ . And  $G'$  is an instance of METRIC TSP, since its weights satisfy the triangle inequality, because of the deep theorem  $2 \leq 1 + 1$ .

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