Combinatorial Optimization – Problem Set 8

You can hand in one of the following problems at the start of Tuesday's problem session. Please explain your solution carefully. Don't forget to put your name.

Matroid Intersection

1. *Show that the problem of finding a Hamilton path from s to t in a given directed graph D can be solved using an intersection of 3 matroids.*

   First remove all edges in $\delta^{\text{in}}(s)$ and $\delta^{\text{out}}(t)$ from D.
   We take the following three matroids with $X = E(D)$:

   $$\begin{aligned} \mathcal{M}_1 &= P(\{\delta^{\text{in}}(v)\}), \\ \mathcal{M}_2 &= P(\{\delta^{\text{out}}(v)\}), \\ \mathcal{M}_3 &= F(G). \end{aligned}$$

   An edge set that is independent with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$ has at each vertex at most one incoming edge and at most one outgoing edge. Hence it consists of cycles and paths. If it is also independent with respect to $\mathcal{M}_3$, then there are no cycles, so it is a disjoint union of paths.
   We claim that D has a Hamilton path if and only if a maximum independent set in this intersection of 3 matroids has $|V(D)| - 1$ edges.
   Let I be a maximum common independent set. If it is a spanning tree, then it must be one single path, which must then be a Hamilton path from s to t (because we removed those edges at the start).
   On the other hand, if I is not spanning tree, then it has at most $|V(D)| - 2$ edges. That means there is no Hamilton cycle, because that would be a common independent set with $|V(D)| - 1$ edges.

2. *Given an undirected graph $G = (V, E)$, an orientation is a directed graph $D = (V, E')$ with a bijection $\varphi : E' \to E$ such that $\varphi(ab) = \{a, b\}$. In other words, each edge $\{a, b\} \in E$ is given a direction, either ab or ba.*
   Given $k : V \to \mathbb{N}$, show that the problem of finding an orientation such that

   $$\delta^{in}(v) = k(v)$$

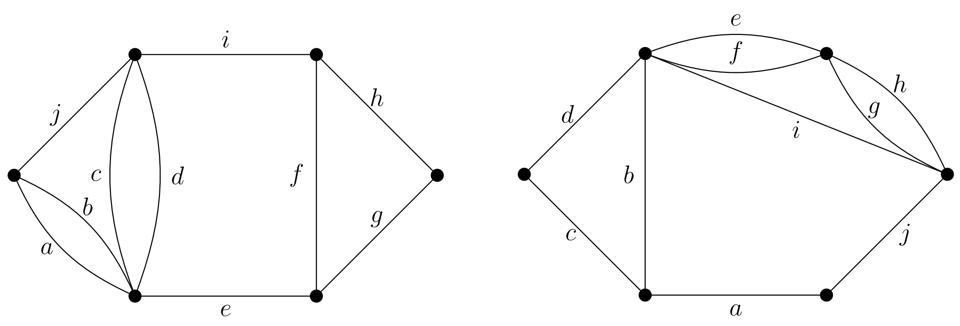
   for each $v \in V$, or showing that none exists, can be solved using the matroid intersection algorithm.

   We just have to give two matroids such that a common independent set is an orientation satisfying that condition. Let X be the set of directed edges ab and ba for each edge $\{a, b\} \in E(G)$. Then the matroids are

   $$\begin{aligned} \mathcal{M}_1 &= P(\{uv, vu\}_{\{u,v\} \in E(G)}), \\ \mathcal{M}_2 &= \{Y \subset X : |Y \cap \delta^{\text{in}}(v)| \leq k \ \forall v \in V(G)\} \end{aligned}$$

   That the second one is a matroid is proved just like for partition matroids.
   A common independent set would be an orientation because of the first matroid (one direction per edge), and it would satisfy $\delta^{\text{in}}(v) \leq k(v)$ for all v due to the second matroid.
   Then an orientation as required exists if and only if there is a common independent set of size $\sum_{v \in V(G)} k(v)$.

3. *Use the matroid intersection algorithm to show that there is no simultaneous spanning tree in the following two graphs (i.e., there is no $T \subset \{a, b, \ldots, j\}$ that is a spanning tree in both).*



If we do the greedy part of the algorithm in alphabetical order, then it would find the common independent set $\{a, c, e, g\}$ (if we used a different order, we would get something "isomorphic"). It is not a spanning tree in either graph. If we now draw the directed graph as in the matroid intersection algorithm, we'll see that there is no path from $X_1 = \{f, h, i\}$ to $X_2 = \{b, d, j\}$.
Alternatively, we could observe that

$$U = \{a, b, c, d, j\}$$

(the set of vertices in D_I from which X_2 can be reached) gives equality in $|I| \leq r_{M_1}(U) + r_{M_2}(X - U)$, which implies that I is maximal (see the matroid intersection theorem).

4. *Make up 2 matroids such that the matroid intersection algorithm needs at least 2 non-greedy steps (i.e. with $|Q| > 1$) to get a maximum common independent set.*