Combinatorial Optimization – Problem Set 4

You can hand in one of the following problems at the start of Tuesday's problem session. Please explain your solution carefully. Don't forget to put your name.

Flows

1. Show that, given an algorithm for finding a maximum set edge-disjoint paths in any directed st-graph, you can use it to find a maximum set of vertex-disjoint directed st-paths in any directed graph.
   Show that there is also a way to do this the other way around.

2. Give an integer program whose optimal solutions correspond to maximum sets of vertex-disjoint st-paths, in a given directed graph G with vertices s, t.
   Give the dual of the relaxation of this program. What objects in the graph do integral dual optimal solutions correspond to?

3. Use flows to give an algorithm for the *binary assignment problem*: Given a bipartite graph G with $c : E(G) \to \mathbb{Z}_{\geq 0}$ and $d : V(G) \to \mathbb{Z}_{\geq 0}$, find a maximum assignment, i.e. a $\varphi : E(G) \to \mathbb{Z}_{\geq 0}$ such that for all edges e we have $\varphi(e) \leq c(e)$ and for all vertices v we have $\sum_{e \in \delta(v)} \varphi(e) \leq d(v)$.

4. A *path flow* g is a flow such that $g(e) > 0$ only on the edges of one directed st-path.
   A *cycle flow* h is a flow such that $h(e) > 0$ only on the edges of one directed cycle.
   Prove that any flow f can be decomposed into path flows and cycle flows, i.e. there is a set $\mathcal{P}$ of path flows and a set $\mathcal{C}$ of cycle flows, such that
   $$f(e) = \sum_{g \in \mathcal{P}} g(e) + \sum_{h \in \mathcal{C}} h(e) \quad \forall e \in E(G).$$

5. Use the first graph below to show that if the augmenting path algorithm chooses the path Q arbitrarily, then its running time is not polynomial.
   Use the second graph below to show that if there are irrational capacities (here $\phi = (\sqrt{5} - 1)/2$, so $\cancel{\phi^2 = \phi - 1}$ $\phi^2 = 1 - \phi$), then the same algorithm may not terminate. Also show that it may not even converge to the right flow value.
   ~~(Hint: Always choose the path with as many horizontal edges as possible. Go through several steps, until you see a pattern.)~~
   The hint is not correct, i.e. that is not the order that leads to non-termination. This question is more annoying than I thought. Consider it highly optional, and just be aware of the existence of such an example. To see a solution with pictures, go to the notes by Jeff Erickson, and there go to lecture 22, second page.