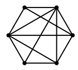
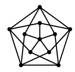
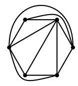
Problem Set 9 – Solutions

Graph Theory 2016 – EPFL – Frank de Zeeuw & Claudiu Valculescu

1. *Determine if the following graphs are planar or not.*

The first one is planar: 

The second graph has $|V(G)| = 11$ and $|E(G)| = 20$. If the graph is planar, then by Euler's formula any planar drawing of it would have $|F(G)| = 2 - |V(G)| + |E(G)| = 2 - 11 + 20 = 11$ faces. But we can see that it contains no triangles, so all faces would be bordered by ≥ 4 edges. Each edge borders 2 faces (there are no cut-edges in this graph), so we have $2e \geq 4f$, hence $40 \geq 44$. Contradiction.

2. *The edges of K_{11} are colored red and blue. Show that the red graph and the blue graph cannot both be planar.*

A planar graph on 11 vertices has at most $3 \cdot 11 - 6 = 27$ edges. So if the red and blue graphs are both planar, we would have at most 54 edges in total. But K_{11} has $\binom{11}{2} = 55$ edges.

3. *Show that if G is triangle-free and planar, then $|E(G)| \leq 2|V(G)| - 4$. Deduce that $K_{3,3}$ is not planar.*

We can assume that G is maximal with these properties, i.e., adding an edge would create a triangle or make it non-planar. Then G is connected. It could be a star (since adding any edge to a star creates a triangle), for which the bound holds. For any tree that is not a star, we can add an edge that does not create a triangle, so we can assume that G is not a tree. Then it has no cut-edges, so every edge bounds 2 faces. It is still not necessarily true that every face is bounded by exactly 4 edges (because in a face with 5 edges, you cannot add an edge without creating a triangle). Howevery, every face is bounded by *at least* 4 edges. Hence we have $2|E(G)| \geq 4|F(G)|$. Plugging into Euler's formula gives

$$2 \leq |V(G)| - |E(G)| + \frac{1}{2}|E(G)| = |V(G)| - \frac{1}{2}|E(G)|,$$

which gives $|E(G)| \leq 2|V(G)| - 4$. Any non-maximal graph with these properties must have fewer edges, so also satisfies the inequality.

$K_{3,3}$ is triangle-free and has 9 edges, so if it were planar we would have $9 \leq 2 \cdot 6 - 4 = 8$, which we don't.

4. *Let G be a planar graph with $\Delta(G) \leq 4$. Show that $\chi(G) \leq 4$.*

We can imitate the proof of the five-colour theorem. Take a vertex V of degree 4 (if there is none it's easy), with neighbors x_1, x_2, x_3, x_4, with edges leaving v in that order. By induction we can color $G - v$ with 4 colors, and we can assume that the color of x_i is i (otherwise we're done). Let R be the set of vertices reachable from x_1 by 13-paths. If $x_3 \notin R$, then we can swap 1 and 3 in R, and then color v with 1. Suppose $x_3 \in R$, so there is a 13-path from x_1 to x_3. Let S be the set of vertices reachable from x_2 by 24-paths. Then $x_4 \notin S$, since a 24-path cannot cross a 13-path. So we can swap 2 and 4 in S, and color v with 2. Done.

5. *Show that K_5 and $K_{3,3}$ can be drawn on a torus without crossings.*

   Use the representation as a square with identified sides. You can pass edges through the sides, which will make it easy to draw these graphs.

   Note: In fact, even K_7 can be drawn on the torus. There is a whole theory of embedding graphs without crossings on surfaces; the situation depends on the *genus* g of the surface. The plane $\mathbb{R}^2$ has genus 0 and the torus has genus 1, and this is the reason that more graphs can be embedded on the torus.

6. *Let C be a closed continuous curve in $\mathbb{R}^2$, which may intersect itself. Show that the regions of $\mathbb{R}^2 \backslash C$ can be colored with two colors, so that two regions do not have the same color if they meet at a boundary arc (a point is not considered a boundary arc).*

   Define a graph with one vertex for each region, and connect two regions if they share a boundary arc, in such a way that the connecting edge intersects C in that boundary arc and nowhere else. This can be done so that we get a planar drawing of a graph. We want to show that this graph is bipartite. It suffices to show that it has no odd cycle.

   Suppose we have a cycle in this graph. Because the drawing is planar, the cycle forms a closed curve D that does not intersect itself, so it has an interior and an exterior. The number of edges in the cycle is the number of times that D intersects with C. We can count these intersections by going along C. If we go along C, we must pass back and forth between the interior and exterior of D, and we must do this an even number of times. Hence $C \cap D$ is even, so the cycle is even.

7. *Consider a finite set of lines in $\mathbb{R}^2$, not all passing through the same point and not all parallel. Show that there is a point where exactly two of the lines intersect.*

   Note: This problem is will not be on the exam, because it requires multigraphs.

   Define a planar multigraph as follows. The vertices are the intersection points of the lines, and on each line we connect two adjacent intersection points by the segment in between them. Add a new point that is connected to the first and last intersection point on every line; to see that this can be done in a planar way, take a large circle containing all intersection points in its interior, place the new point outside that circle, connect the new point with all the points where the circle is hit by a line, and continue these edges along the lines to the extremal points. Note that in this way we may get multiple edges, because a point may be extremal on more than one line, and then it will be connected to the special point by more than one edge.

   The results that we have seen for planar graphs hold equally well for planar multigraphs. In particular, the induction proof of Euler's formula still works, since if we remove one of multiple edges between two vertices, then it still increases the number of faces by one.

   Every vertex in this multigraph has even degree, since at an intersection point every line continues in two directions (and the added point has two edges for every line, but this doesn't matter). Because the multigraph is planar, the multigraph version of a fact from class gives a vertex of degree at most 5. Then this must be a vertex of degree 4, which means it is a point where exactly two lines intersect (it is not the added point, since its degree is twice the number of lines).

   Note: This is the dual of the Sylvester-Gallai Theorem. By "dual" we mean the duality of the projective plane. The Sylvester-Gallai Theorem states that for every finite set of non-collinear points, there is a line that passes through exactly two of the points.