Problem Set 8 – Solutions

Graph Theory 2016 – EPFL – Frank de Zeeuw & Claudiu Valculescu

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1. *Let $k \geq 2$. Show that every k-connected graph with at least $2k$ vertices contains a cycle of length at least $2k$.*

   If G is k-connected, then $\delta(G) \geq k$, so by Proposition 1.2.2, G contains a cycle of length at least $\delta(G) + 1 \geq k$. Let C be a longest cycle in G, and suppose its length is less than $2k$. So we have $k \leq |V(C)| < 2k$. Since $|V(G)| \geq 2k$, there is a vertex $x \notin V(C)$.

   By Lemma 8.2.2 from the notes (or by a similar application of Menger's Theorem), and using the fact that $|V(C)| \geq k$, there are k paths from x to $V(C)$ that are disjoint except for x, with each path containing only one vertex of C. By the pigeonhole principle and the fact that $|V(C)| < 2k$, there must be two of these paths that end at adjacent vertices y, z of $V(C)$. Then replacing the edge yz in C by the path from y to x to z, we get a new cycle. It is longer, since each of the two paths has at least one edge. This contradicts the choice of C.

2. *Show that any two vertices in a 3-connected graph are connected by two internally disjoint paths of different lengths.*

   Let x, y be two vertices in a 3-connected graph. There are three internally disjoint paths P, Q, R between x and y; if two of them have different lengths we are done, so assume P, Q, R have the same length L. At most one of them could be a single edge, so we have $L \geq 2$. Hence $V(P)\backslash\{x, y\}$ and $V(Q)\backslash\{x, y\}$ are not empty.

   Let p and q be two vertices on two of these paths, not equal to x or y. There are three internally disjoint paths from p to q; at least one of these paths does not pass through x or y. So for any p from one of P, Q, R, and q from one of the other two, there is a path between p and q that does not pass through x or y. Let S be the shortest path between any such p and q; without loss of generality, S goes from $r \in V(P)\backslash\{x, y\}$ to $s \in V(Q)\backslash\{x, y\}$. By minimality, S does not contain any vertex of P or Q other than r and s, and S does not contain any vertex of R.

   We now have two new paths from x to y that are internally disjoint from R: One goes from x along P to r, then along S to s, and then along Q to y; the other goes from x along Q to s, along S to r, and along P to y. The total length of these two paths is $2L + 2|E(S)|$, so one of the two must have length different from L.

3. *Prove that a k-connected graph G has $|E(G)| \geq \frac{1}{2}k|V(G)|$. For even k, find a k-connected graph G with $|E(G)| = \frac{1}{2}k|V(G)|$.*

   The lower bound follows from $\delta(G) \geq k$, since $|E(G)| = \frac{1}{2}\sum d(v) \geq \frac{1}{2}k|V(G)|$.

   For the construction, start with a cycle and connect each vertex to the $k/2$ nearest neighbors on each side. That gives the right number of edges. To see that it is k-connected, remove $k-1$ vertices and consider two vertices x, y. One of the two half-cycles between x and y has less than $k/2$ removed vertices. Then along that half-cycle we can find a path from x to y, because if we travel in one direction, every vertex has $k/2$ neighbors in that direction, not all of which can have been removed.

4. *A graph G is k-edge-connected if for every $S \subset E(G)$ of size $k-1$ the graph $G - S$ is connected. Show that if G is k-connected, then G is k-edge-connected. Give an example to show that the converse is not true.*

   Suppose G is k-connected, and let F be a minimal set of edges such that $G - F$ is disconnected. If some vertex v of G is not incident to F, then let C be the component of $G - F$ that contains v. Every edge of F has at most one vertex in C, by minimality of F. Taking all the endpoints of edges of F that lie in C, we get at most $|F|$ vertices that disconnect v from some other vertex. So $|F| > k - 1$.

   Otherwise, every vertex is incident with an edge of F. For any vertex v, $N(v)$ disconnects the graph (unless it is complete, in which case it is k-edge-connected). Every $w \in N(v)$ is either connected to v by an edge of F, or is incident to a distinct edge of F (if for $w, w' \in N(v)$ we have $ww' \in F$, it would contradict minimality of F). So $k - 1 < |N(v)| \leq |F|$.

   Take two complete graphs sharing one vertex. It is highly edge-connected but only 1-connected.

5. *Use Menger's Theorem to reprove König's Theorem.*

   Let G be a bipartite graph with bipartition $V(G) = A \cup B$. A matching of size m corresponds to a set of m disjoint AB-paths (in the sense of Menger's Theorem). A vertex cover corresponds to a set of vertices that cover every original edge, which means that it cuts every AB-path, so it is an AB-separator. By Menger's Theorem, the maximum size of a set of disjoint AB-paths equals the minimum size of an AB-separator.

   If you prefer the version of Menger with two vertices, add a vertex a connected to all vertices in A, and a vertex b connected to all vertices in B. Then a matching corresponds to a set of internally disjoint ab-paths.

6. *Show that in a 3-connected graph, any two longest cycles share at least 3 vertices.*

   Suppose C_1 and C_2 are two longest cycles (it doesn't matter if we interpret this as both being the same length, or as one being longest and the other second-longest).

   If C_1 and C_2 are disjoint, then Menger's Theorem gives three disjoint paths between $V(C_1)$ and $V(C_2)$. If C_1 and C_2 share exactly one vertex x, then Menger gives two disjoint paths between $V(C_1)$ and $V(C_2)$ (not counting the one-vertex path x). Either way, we have two disjoint paths P, Q between C_1 and C_2; let's say P goes from $p_1 \in V(C_1)$ to $p_2 \in V(C_2)$, and that Q goes from $q_1 \in V(C_1)$ to $q_2 \in V(C_2)$.

   In the case where C_1 and C_2 are disjoint, we get a longer cycle by going from p_1 to q_1 along the longer part of C_1, from q_1 to q_2 along Q, from q_2 to p_2 along the longer part of C_2, and finally from p_2 to p_1 along P.

   When C_1 and C_2 share one vertex x, this does not work, and we do the following. Let C_3 be the following cycle: Go from p_1 to q_1 along the part of C_1 not containing x, from q_1 to q_2 along Q, from q_2 to p_2 along the part of C_2 containing x, and finally from p_2 back to p_1 along P. Let C_4 be the following cycle: Go from p_2 to q_2 along the part of C_2 not containing x, from q_2 to q_1 along Q, from q_1 to p_1 along the part of C_1 containing x, and finally from p_1 back to p_2 along P. Now we have $|E(C_3)| + |E(C_4)| = |E(C_1)| + |E(C_2)| + |E(P)| + |E(Q)|$, so one of C_3, C_4 is longer than C_1 or C_2.

   Now suppose C_1 and C_2 share exactly two vertices. Menger gives one path P from $V(C_1)$ to $V(C_2)$. We won't spell it out, but again we can find two new cycles C_3, C_4 with $|E(C_3)| + |E(C_4)| = |E(C_1)| + |E(C_2)| + |E(P)|$, which shows that one of C_3, C_4 must be longer than C_1 or C_2.

*7. *Prove that a graph G with $|E(G)| \geq 2k|V(G)|$ contains a k-connected subgraph.*

We use induction on $|V(G)|$ with the following strengthened induction claim:

If $|V(G)| \geq 2k$ and $|E(G)| \geq 2k|V(G)| - 2k^2$, then G has a k-connected subgraph.

This finishes the problem, since if $|E(G)| \geq 2k|V(G)|$, then $\binom{|V(G)|}{2} \geq 2k|V(G)|$, which implies $|V(G)| \geq 4k+1 > 2k$, so the induction claim applies.

First note that if $|V(G)| = 2k$, then $2k|V(G)| - 2k^2 = 2k^2 > \binom{2k}{2} = \binom{|V(G)|}{2}$, so we cannot have this many edges, and the claim trivially holds. So suppose we have G with $|V(G)| > 2k$ and $|E(G)| \geq 2k|V(G)| - 2k^2$.

If G has a vertex v of degree less than $2k$, then we can remove v and apply induction, since

$$\begin{aligned} |E(G-v)| &> |E(G)| - 2k \\ &\geq 2k|V(G)| - 2k^2 - 2k \\ &= 2k(|V(G)| - 1) - 2k^2 \\ &= 2k|V(G-v)| - 2k^2. \end{aligned}$$

Thus we can assume that every vertex of G has degree at least $2k$.

If G is k-connected, then we are done. Otherwise, there is a set X with $|X| = k-1$ that disconnects G. Thus we can get two subgraphs G_1 and G_2 with $V(G_1) \cup V(G_2) = V(G)$, $V(G_1) \cap V(G_2) = X$, and $E(G_1) \cup E(G_2) = E(G)$. For instance, let G_1 be the union of X with a component of $G - X$, and let G_2 be the union of X with the other components of $G - X$.

Since each G_i contains a vertex not in X, and that vertex has degree at least $2k$, we have $|V(G_i)| \geq 2k$ for both i. We cannot have $|E(G_i)| < 2k|V(G_i)| - 2k^2$ for both i, since then

$$\begin{aligned} |E(G)| &\leq |E(G_1)| + |E(G_2)| \\ &< (2k|V(G_1)| - 2k^2) + (2k|V(G_2)| - 2k^2) \\ &= 2k(|V(G_1)| + |V(G_2)| - 2k) \\ &= 2k(|V(G)| + k - 1 - 2k) \\ &< 2k|V(G)| - 2k^2. \end{aligned}$$

Thus the induction claim applies to at least one of the G_i, which gives a k-connected subgraph.