

Problem Set 7 – Solutions

Graph Theory 2016 – EPFL – Frank de Zeeuw & Claudiu Valculescu

1. Show that a 2-connected graph G is 2-edge-connected, i.e., there is no edge $e \in E(G)$ such that $G - e$ is disconnected.

If G is not 2-edge-connected, then there is an edge xy such that $G - xy$ is disconnected. Note that $G - xy$ has exactly two components. If both components have more than one vertex, then both x and y are cut-vertices. Indeed, a path in G between the components must pass through xy , so removing x or y breaks such a path.

There is an exception, if one of the two components contains only one vertex (x or y). In that case that vertex is not a cut-vertex, but the other one is.

In both cases, G not being 2-edge-connected implies there is a cut-vertex, so G is not 2-connected.

2. Show that if G is 2-connected, then for every two edges there is a cycle containing both those edges. Is the converse true?

Let uv and xy be any two edges (they may be disjoint or share a vertex). Create a new vertex w attached to u and v , and a new vertex z attached to x and y . This graph is still 2-connected. By a fact from the lecture, there is a cycle containing w and z , which must contain the paths uwv and xzy , so replacing those paths by uv and xy gives a cycle containing uv and xy .

The converse is true. We can for instance prove that every two vertices are connected by two internally disjoint paths. Take any two edges containing the two given vertices. There is a cycle containing those two edges, hence the cycle also contains the two given vertices, so it gives two internally disjoint paths between them.

3. Prove that a graph is 2-connected if and only if for every three vertices x, y, z , there is a path from x to z that passes through y .

if: Consider $G - z$ for $z \in V(G)$, and take any $x, y \in V(G - z)$. The condition gives that there is a path P in G from x to z through y . This path has a subpath from x to y , which does not pass through z , since otherwise P would not be a path. Thus there is a path from x to y in $G - z$, which means that $G - z$ is connected. The fact that this is true for every $z \in V(G)$ means that G is 2-connected.

only if: Assume G is 2-connected. Add a vertex w that is connected to both x and z ; the resulting graph is still 2-connected. By Whitney's Theorem, there are two internally disjoint paths from w to y . Since x and z are the only neighbors of w , one of the paths must pass through x and the other must pass through z . Remove w and take the path from x to y , followed by the path from y to z . Since the two paths were internally disjoint, this really is a path.

4. Let G be a connected graph with $|V(G)| \geq 3$. Create a new graph G' by adding the edge xy between every $x, y \in V(G)$ such that $d_G(x, y) = 2$. Show that G' is 2-connected.

If G' is not 2-connected, then there is a cut-vertex v such that $G' - v$ is disconnected. We claim that v has two G -neighbors x, y that are in different components of $G' - v$. This would be a contradiction, since then $d_G(x, y) = 2$ and we should have added xy in G' , or $d_G(x, y) = 1$ and xy is an edge in G ; either ways x and y would be in the same component of $G' - v$.

We prove the stronger claim that every component of $G' - v$ contains a G -neighbor of v . Every component of $G' - v$ must contain a G' -neighbor u of v . If u is also a G -neighbor of v , we are done. Otherwise, the edge uv must have been added in G' , so $d_G(u, v) = 2$. Then there is a vertex w such that $uw, vw \in E(G)$, which implies that w is in the same component of $G' - v$ as u , and it is a G -neighbor of v .

5. *Find the smallest 3-regular connected graph that is not 2-connected.*

It is the “double fly-swatter”: Take two K_4 s, for each pick an edge and subdivide it (replace it by a P_2 , with endpoints at the vertices of the edge), then connect the two midpoints of the P_2 s.

We show that such a graph must have at least 10 vertices, which means the double fly-swatter is optimal. There must be a cut-vertex x . It must have one neighbor in one component of $G - x$ and two in the other (or one neighbor in three components, but that will be worse). In the component with two neighbors u, v , these neighbors must have at least one further neighbor s . But s must have three neighbors, which can include u, v but not x , so this component of $G - x$ must have at least four vertices. The other component of $G - x$ has one neighbor a of x , which must have two further neighbors b, c . By the same argument as before, $G - a$ must have four vertices in this component. Altogether we have now counted at least 10 vertices.

- *6. *Prove that any vertex x in a 2-connected graph G has a neighbor y such that $G - x - y$ is connected.*

Suppose that for every $y \in N(x)$, $G - x - y$ is disconnected. Let y^* be the neighbor of x for which $G - x - y^*$ has the smallest component H (i.e., H has the minimum number of vertices among all components of $G - x - y$ for all $y \in N(x)$). Let K_1, \dots, K_m be the other components of $G - x - y^*$.

Observe that $G - x$ is connected, so each of H, K_1, \dots, K_m has a path to y^* not passing through x . Similarly, each of H, K_1, \dots, K_m has a path to x not passing through y^* .

In particular, H must contain a neighbor z of x . By assumption, $G - x - z$ is also disconnected. We claim that $G - x - z$ has a component that is contained in $H - z$, which would make it smaller than H , contradicting the choice of y^* .

Suppose that $G - x - z$ does not have a component contained in $H - z$, so every vertex in H is connected by a path in $G - x - z$ to a vertex outside of $H - z$. Such a path must go through y^* , since otherwise H would not be a component of $G - x - y^*$. But then $G - x - z$ would be connected, because every K_i is connected to y^* by a path that does not pass through x or z . This is a contradiction.
