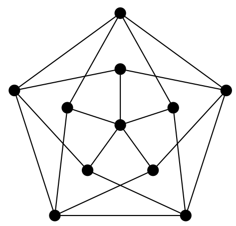
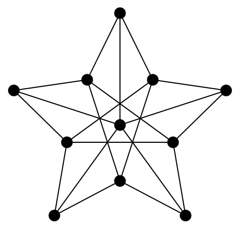
Problem Set 6 – Solutions

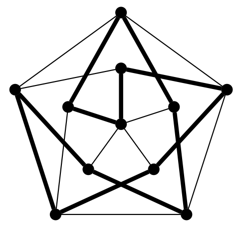
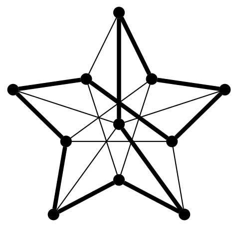
Graph Theory 2016 – EPFL – Frank de Zeeuw & Claudiu Valculescu

1. *Determine the girth and circumference of the following graphs. In particular, do they have Hamilton cycles?*

The graph on the left has girth 4; it's easy to find a 4-cycle and see that there is no 3-cycle. It has circumference 11, since below is an 11-cycle (a Hamilton cycle).

The graph on the right also has girth 4. It also has circumference 11, since below is an 11-cycle.

2. *Determine the circumference of the complete bipartite graph $K_{m,n}$ for all m, n.*

The circumference is $2 \cdot \min\{m, n\}$. Any cycle must be even, and it must alternate vertices from the two sides. Thus it is longest when it passes through all the vertices on the smaller side.

3. *Show that a k-regular graph with girth 4 must have at least $2k$ vertices.*

Take one vertex and its k neighbors, together $k + 1$ vertices. These neighbors are not connected, since otherwise there would be a triangle. Thus, if we look at one of these neighbors, it has $k-1$ neighbors that are not among the $k+1$ we counted so far. Altogether we have $1 + k + (k - 1) = 2k$ vertices.

Alternatively, take two adjacent vertices. They have no common neighbors, since otherwise there would be a triangle. Each has $k - 1$ new neighbors, so altogether we have $2 + 2(k - 1) = 2k$ vertices.

4. *Let G be a 3-regular graph with $\chi_e(G) = 4$. Prove that G does not have a Hamilton cycle.*

Note that since $\sum d(v) = 2|E(G)|$ and every $d(v) = 3$, G must have an even number of vertices. Suppose that G has a Hamilton cycle C. It must be even, so we can color it with 2 colors. Every vertex has 2 edges from C and one other edge, so the edges not in C form a matching. Hence we can color these edges with one color. This gives an edge-coloring of G with 3 colors, contradicting $\chi_e(G) = 4$.

5. *Show that the proof of Dirac's Theorem also proves the following statement: If for all non-adjacent $u, v \in V(G)$ we have $d(u) + d(v) \geq |V(G)|$, then G has a Hamilton cycle.*

   There were two places in the proof of Dirac's Theorem where we used the condition that $\delta(G) \geq \frac{1}{2}|V(G)|$: To show that the two sets $|\{x_i : 1 \leq i \leq k-1, x_i x_k \in E(G)\}|$ and $|\{x_i : 1 \leq i \leq k-1, x_i x_k \in E(G)\}|$ intersect, and to show that G is connected. The first part follows directly from the assumption $d(u) + d(v) \geq |V(G)|$. For the second part, suppose G has at least 2 components. Then a vertex u from one component is not adjacent to a vertex v from another component, so we have $d(u) + d(v) \geq |V(G)|$. But the component containing u has at least $d(u) + 1$ vertices, and the component containing v has at least $d(v) + 1$, which gives more than $|V(G)|$ vertices in total.

6. *Use Problem 5 to give a short proof of the fact that a graph G with $|E(G)| > \binom{|V(G)|-1}{2} + 1$ has a Hamilton cycle.*

   Set $|V(G)| = n$. If the graph is complete then it has a Hamilton cycle. Consider two non-adjacent vertices u, v. Remove them to get a graph $G - u - v$ with $n-2$ vertices and $|E(G)| - d(u) - d(v)$ edges. It has at most $\binom{n-2}{2}$ edges, so we have

$$\binom{n-2}{2} \geq |E(G)| - d(u) - d(v) > \binom{n-1}{2} + 1 - d(u) - d(v)$$

$$\Longrightarrow \quad d(u) + d(v) > \binom{n-1}{2} - \binom{n-2}{2} + 1 = n - 1.$$

   Thus the condition of Problem 5 holds, so G has a Hamilton cycle.

*7. *Show that if $|E(G)| \geq |V(G)| + 1$, then the girth of G is smaller than $\frac{2}{3}|V(G)| + 1$.*

   We can assume that G is connected, since otherwise one of its components H must have $|E(H)| \geq |V(H)| + 1$, so we could work with H to find an even smaller cycle. Thus we can think of G as consisting of a spanning tree with (at least) two edges added. Each added edge creates one cycle.

   Add one edge to the spanning tree, which creates exactly one cycle C. If C is shorter than $\frac{2}{3}|V(G)| + 1$ we are done, so we can assume that $|E(C)| \geq \frac{2}{3}|V(G)| + 1$. Remove the edges of C; the remainder is a forest. Each component of the forest contains exactly one vertex of C: If it contained 0, G would not be connected, and if it contained at least 2, then there would be more than one cycle after adding the edge to the spanning tree. Also note that the forest has at most $|V(G)| - (\frac{2}{3}|V(G)| + 1) = \frac{1}{3}|V(G)| - 1$ edges.

   Now add the second edge. It either creates a cycle in a component of the forest, or it connects two components of the forest. In the first case, the created cycle has length at most $\frac{1}{3}|V(G)|$, and we would be done. In the second case, we obtain a path P between two vertices x, y of C, which has length at most $|V(G)| + 1 - |E(C)|$. The shortest path along C between x and y has length at most $\frac{1}{2}|E(C)|$, so together with P it gives a cycle of length at most

$$|V(G)| + 1 - |E(C)| + \frac{1}{2}|E(C)| = |V(G)| + 1 - \frac{1}{2}|E(C)|$$

$$\leq |V(G)| + 1 - \frac{1}{2}\left(\frac{2}{3}|V(G)| + 1\right) = \frac{2}{3}|V(G)| + \frac{1}{2} < \frac{2}{3}|V(G)| + 1.$$

   Note: By being slightly more careful we can get the bound $\lfloor \frac{2}{3}|V(G)| + \frac{2}{3} \rfloor$. This is best possible, because it is what we get for two vertices connected by three vertex-disjoint paths, with those paths as close in length as possible (the calculation depends on $|V(G)|$ modulo 3).

*8. *Show that if G contains no even cycle, then $|E(G)| \le \frac{3}{2}|V(G)|$.*

Let G be a graph without even cycles. We can assume that it is connected, since if the bound holds for the connected components, then it carries over to the whole graph.

The key observation is that for any cycle in G, between two vertices on the cycle there is no path other than the two paths given by the cycle.

More precisely, let $C = x_1 \cdots x_k x_1$ be an odd cycle, and suppose there is a third path between two vertices of the cycle. Let $P = x_i y_1 \cdots y_m x_j$ be the shortest path such path. Then the vertices y_i are not on C, otherwise there is a shorter such path. Of the two paths between x_i and x_j along the cycle, one must be odd and one must be even. If m is even, P gives an even cycle together with the even path along the cycle; if m is odd, P gives an even cycle together with the odd path along the cycle. Hence there are no such "chord-paths".

This implies that any two cycles shares at most one vertex, since otherwise a part of one cycle would give a chord-path on the other cycle. In particular, each edge is contained in at most one cycle.

Let c be the number of cycles in the graph. Then $|E(G)| \ge 3c$, since each cycle has at least 3 edges. On the other hand, if we remove one edge from each cycle, we have a forest, so $|E(G)| - c \le |V(G)| - 1$. Combining these inequalities gives

$$|E(G)| \le c + |V(G)| - 1 \le \frac{|E(G)|}{3} + |V(G)| - 1 \implies |E(G)| \le \frac{3}{2}(|V(G)| - 1) \le \frac{3}{2}|V(G)|.$$

Note: The bound $|E(G)| \le \frac{3}{2}(|V(G)| - 1)$ is essentially tight. For $|V(G)| = 2m + 1$ odd, take the graph with m triangles that all have the same vertex in common. It has $3m = 3 \cdot (|V(G)| - 1)/2$ edges.

For $|V(G)| = 2m$ is even, take $m - 1$ triangles with one point in common, and add a leaf anywhere. This graph has $3(m-1) + 1 = 3m - 2 = \frac{3}{2}|V(G)| - 2 = \lfloor \frac{3}{2}(|V(G)| - 1) \rfloor$ edges, which is best possible.