Problem Set 5 – Solutions

Graph Theory 2016 – EPFL – Frank de Zeeuw & Claudiu Valculescu

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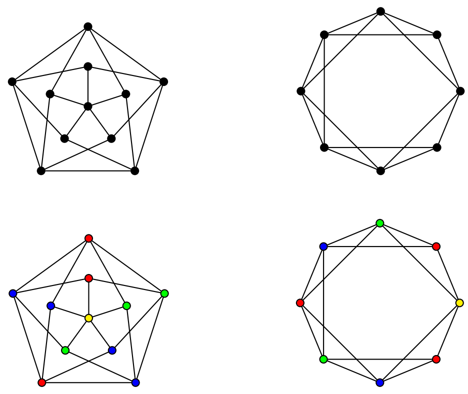
1. *Determine the chromatic number of the two graphs below.*



Both have chromatic number 4. Shown are 4-colorings for both.

To show that the coloring of the first graph is optimal, we try to 3-color it. Start with the outer C_5: up to isomorphism there is only one coloring, red-blue-red-blue-green. This forces a red, blue, and green vertex on the inner ring of 5 vertices, which forces a fourth color on the middle vertex.

Let's try to 3-color the second graph. Any triangle in the graph needs exactly 3 colors. That triangle has a "neighboring" triangle, with which it shares two vertices. The 3 colors of the first triangle force the color of the third vertex of the neighboring triangle. Continuing like that, we can go around forcing all the vertices, until we end up back at the first triangle, where we'll have a conflict. So we need at least 4 colors.

2. *Determine the chromatic number of the following graph: Given a finite set of lines in $\mathbb{R}^2$ with no 2 parallel and no 3 concurrent, let the vertices be the intersection points of the lines, and connect two intersection points if they are consecutive on one of the lines.*

We claim that $\chi(G) = 3$. Clearly $\chi(G) \geq 3$, since three lines give a triangle, which cannot be 2-colored. So we need to show that a 3-coloring exists.

Rotate the plane so that none of the intersection points have the same y-coordinate; this is possible, since there is only a finite number of intersection points, but infinitely many possible rotations.

Now imagine sweeping the points with a vertical line, moving that line from far left to far right, and coloring the vertices in the order that we encounter them on the vertical line. Because of the rotation, we never encounter two vertices at the same time.

We can color with 3 colors this way, because whenever we encounter a vertex, at most 2 of its neighbors have been colored before. To see why, observe that a vertex has at most 4 neighbors, since no 3 given lines intersect in the same point, and a vertex has at most 2 neighbors on each of the lines going through it. Moreover, at most 2 of these neighbors are on the same side of the vertical line that we sweep with.

Alternative formulations: We could also take a line in a random direction and sweep with that; then we don't have to rotate beforehand. Or, instead of sweeping with a vertical line, we could just say that we order the vertices by increasing x-coordinate, and we apply the greedy algorithm in that order.

3. *Prove that if every two odd cycles of G intersect in at least one vertex, then $\chi(G) \leq 5$.*

   If G has no odd cycles, then G is bipartite by a proposition from the first lecture, which means that $\chi(G) \leq 2$. Thus we can assume that G has at least one odd cycle.

   Let C be any odd cycle, and remove its vertices from G to get a new graph $G - C$. It has no odd cycles, since every odd cycle previously intersected C. This implies that $G - C$ is bipartite, or in other words 2-colorable. Then we can combine a 2-coloring of $G - C$ with a 3-coloring of C to get a 5-coloring of G.

4. *Determine the edge-chromatic numbers of the complete graphs K_n.*

   $\chi_e(K_{2k-1}) = 2k - 1$: To get a $2k - 1$-coloring, place the vertices v_i on a circle with equal spacing. Then for each vertex v_i, give the same color to the edges $v_{i-1}v_{i+1}$, $v_{i-2}v_{i+2}$, etc. (these edges will be parallel). This way we color all the edges with $2k - 1$ colors.

   Suppose we could color the edges of K_{2k-1} with $2k - 2$ colors. Each color class has at most $k - 1$ edges, so with $2k - 2$ colors we can color at most $(2k - 2)(k - 1)$ edges. But K_{2k-1} has $\binom{2k-1}{2} = (2k - 1)(k - 1)$ edges, so this can't work.

   $\chi_e(K_{2k}) = 2k - 1$: Now place $2k - 1$ of the vertices v_i on a circle with equal spacing, and put the remaining vertex u at the center of the circle. Then for each v_i, color in the same way as in the odd case, and also give that color to the edge uv_i. This gives an edge coloring with $2k - 1$ colors.

   In this case we have $\Delta(K_{2k}) = 2k - 1$, so by a theorem from class there is no edge coloring with fewer colors.

5. *Prove that if G is bipartite, then $\chi_e(G) = \Delta(G)$.*

   If G is bipartite and k-regular, then by Hall's Theorem, it has a perfect matching M. Removing M gives a $(k - 1)$-regular bipartite graph G'. By induction, we can color G' with $\Delta(G') = k - 1$ colors, and use a kth color for the edges of M.

   If G is not regular, we can make it $\Delta(G)$-regular and apply the above. First add vertices on the smaller side of G so that the two sides have the same size. As long as the graph is not regular, there must be a vertex of degree less than $\Delta(G)$ on both sides, so we can connect those. Continuing like this we get a $\Delta(G)$-regular bipartite graph.

*6. *Find the smallest 3-regular graph that contains no C_3 or C_4, and prove that it is the only graph of that size with those properties.*

   The Petersen graph has these properties, and we will show that any graph with these properties has at least 10 vertices, and any graph on 10 vertices with these properties must be the Petersen graph.

   Start with any vertex 1. It has three neighbors $2, 3, 4$. Since there is no C_3, no two of $2, 3, 4$ are adjacent, and since there is no C_4, no two share a neighbor other than 1. Thus each one has two more neighbors; let's call them $2a$ and $2b$, $3a$ and $3b$, and $4a$ and $4b$. We now have a tree on 10 vertices that must be contained in any 3-regular graph without C_3 or C_4. This already proves that any such graph has at least 10 vertices.

   The vertices $1, 2, 3, 4$ already have all their neighbors, so it remains to add edges between $2a, 2b, 3a, 3b, 4a, 4b$; each must get two more edges. The a and b vertices belonging to the same vertex from $2, 3, 4$ cannot be adjacent, since that would create a C_3. An a or b vertex cannot be adjacent to both vertices of an a-b pair, since that would create a C_4. Without loss of generality, we can assume that $2a$ is adjacent to $3a$ and $4a$, and $2b$ is adjacent to $3b$ and $4b$. Then $3a$ must be adjacent to $4b$, and $3b$ must be adjacent to $4a$.

This shows that any 3-regular graph on 10 vertices without C_3 or C_4 is isomorphic to the current graph.

Since the Petersen graph is such a graph, it must be this one. More concretely, we can see that $1, 2, 2a, 3a, 3$ form a 5-cycle, and $4, 4a, 3b, 2b, 4b$ form a disjoint 5-cycle. There are 5 edges left, and if two of those edges are incident to adjacent vertices on one 5-cycle, then they are incident to non-adjacent vertices on the other 5-cycle.

*7. *Show that a connected graph with $2k$ edges can be decomposed into k paths of length two.*

It is natural to apply induction, by finding a P_2 and removing it. The problem is that after the removal the components may have odd size, in which case the induction statement does not apply.

Let $P = xyz$ be a path of length two in the given graph G. Note that $G - xy - yz$ has at most three connected components, since removing an edge from a graph creates at most one more component. If all connected components of $G-xy-yz$ have an even number of edges, then we are done by induction. Otherwise, $G - xy - yz$ has two components with an odd number of edges (and possibly one more with an even number, to which we can separately apply induction). Each odd component is attached to at least one of x, y, z, and they are not attached to the same vertex. Without loss of generality, we can assume that the odd component C_1 contains x and the odd component C_2 contains y or z. Then we can add xy to C_1 and yz to C_2 to get two connected subgraphs with an even number of edges. Both have fewer edges than G, so we can apply induction to each.

Alternative solution: Take a longest path $P = x_1x_2x_3 \dots x_k$. The neighbors of x_1 must be on P, so $G - x_1x_2$ is connected. If the neighbors of x_2 are also all on P, then $G - x_1x_2 - x_2x_3$ is connected, so we can remove the path $x_1x_2x_3$ and apply induction to $G - x_1x_2 - x_2x_3$. Otherwise, x_2 has a neighbor y not on the path. Then y does not have neighbors off the path, since otherwise the path could be extended. Thus we can remove the path x_1x_2y and apply induction to the connected subgraph $G - x_1x_2 - x_2y$.

*8. *Show that any graph G with $|V(G)| \geq 4$ and $|E(G)| \geq 2|V(G)| - 3$ contains two cycles of the same length.*

Let n be the number of vertices, and $f(n)$ the minimum number of edges that guarantee two cycles of the same length.

Take a spanning tree for the graph, which has $n-1$ edges. Every added edge creates at least one cycle, whose length is from the set $\{3, \dots, n\}$ of size $n-2$. Thus, if we add $n-1$ edges, two of the created cycles must have the same length. This shows $f(n) \leq 2n - 2$.

We can do better. If there is no cycle of length n, then each cycle has length in the set $\{3, \dots, n-1\}$ of size $n-3$, so adding $n-2$ edges to the $n-1$ of the tree implies two cycles have the same length. If there is a cycle of length n (a Hamilton cycle), then adding any edge to the n edges of this cycle creates *two* new cycles, with lengths from the set $\{3, \dots, n-1\}$ of size $n-3$. Thus, if in this case we add $n-3$ edges to the n of the Hamilton cycle, then since $2n - 6 > n - 3$ (for $n \geq 4$), we get two cycles of the same length. This shows $f(n) \leq 2n - 3$, so $|E(G)| \geq 2|V(G)| - 2$ implies two cycles of the same length.

This is not best possible, and it is an open problem to determine $f(n)$ precisely. The best known upper bound is $f(n) < n + 1.98\sqrt{n}$. See for instance the survey "Some open problems on cycles" by Chunhui Lai and Mingjing Liu.

*9. *Prove that a graph G with $|E(G)| \geq |V(G)| + 4$ contains two edge-disjoint cycles.*

If G contains a C_3 or C_4, then removing the at most 4 edges of this cycle gives a graph with $|V(G)|$ edges, and this graph has a cycle, which is edge-disjoint from the first cycle.

If G has a vertex of degree at most 1, then we can remove it and apply induction. If G has a vertex y of degree 2, adjacent to x and z, then we can remove y and its two edges, and replace them by the edge xz (which was not in G, since that would give a C_3). We have one fewer vertex and one fewer edge, so we can apply induction to get two disjoint cycles. Removing xz and returning xy and yz leaves these cycles intact and disjoint. The base case of the induction is $|V(G)| = 5$ (the smallest case where we can have $|E(G)| \geq |V(G)| + 4$), which is easy to check.

By the above we can assume that G has no C_3 or C_4, and every vertex has degree at least 3. By the same argument as in Problem 6, such a graph has at least 10 vertices. Suppose that $|E(G)| = |V(G)| + 4$. Then we have

$$14 = |V(G)| + 4 = |E(G)| = \frac{1}{2}\sum d(v) \geq \frac{1}{2} \cdot 3 \cdot |V(G)| = 15.$$

This contradiction implies that if $|E(G)| = |V(G)| + 4$, then G has two edge-disjoint cycles. Then the same is true for any G with $|E(G)| > |V(G)| + 4$, by removing edges.

*10. *Let G be a graph with the property that any subgraph H has $|E(H)| \leq 2|V(H)| - 2$. Prove that G can be decomposed into two forests. More precisely, show that G contains forests F_1, F_2 with $E(F_1) \cap E(F_2) = \emptyset$ and $E(F_1) \cup E(F_2) = E(G)$.*

First observe the following. If we have a proper subgraph H with $|E(H)| = 2|V(H)| - 2$, then we can do the following. Decompose H into two forests using induction. Any vertex $v \in V(G)\backslash V(H)$ has at most two edges to H, since otherwise the subgraph $H + v$ (adding v and all edges between v and H) would have too many edges. If there is a vertex $v \in V(G)\backslash V(H)$ with exactly two edges to H, then $H + v$ can also be decomposed into two forests, by adding one edge to each forest of H. By repeating this, we can assume that we have a subgraph H' such that $|E(H')| = 2|V(H')| - 2$, which is decomposed into two forests, and has the property that any $v \in V(G)\backslash V(H')$ has at most one edge to H'. Because of this property, we can "contract" H': We remove it, replace it by a single vertex h, and for any edge from $v \in V(G)\backslash V(H')$ to H we add an edge from v to h. By induction, the contracted graph can be decomposed into two forests. These two forests can be combined with the two forests of H' into a decomposition of G into two forests.

Therefore, we can assume that G has no proper subgraph H with $|E(H)| = 2|V(H)| - 2$.

Now we start from a different direction. The graph G itself has $|E(G)| \leq 2|V(G)| - 2$, so $\sum d(v) = 2|E(G)| < 4|V(G)|$ implies that G has a vertex x of degree at most 3. If $d(x) \leq 2$, then we can apply induction to $G - x$ to decompose $G - x$ into two forests. If $d(x) = 0$ we are done. If $d(x) = 1$ then we add the edge incident to x to either forest, and if $d(x) = 2$ then we add one edge to each forest; either way no cycles are created.

We can assume that $d(x) = 3$. Its three neighbors cannot be pairwise adjacent, because then these four vertices would form a K_4, which would be a subgraph H with $|E(H)| = 2|V(H)| - 2$. Thus x has two neighbors a, b that are not adjacent to each other, and another neighbor c.

Now consider $G - x + ab$. Note that adding one edge to $G - x$ does not break the subgraph condition, because G had $|E(H)| < 2|V(H)| - 2$ for every proper subgraph H. Now we again use induction to decompose $G - x + ab$ into two forests. We remove ab and add ax and bx to the forest that ab was, which does not create a cycle. We add cx to the other forest, which also does not create a forest. Then we have decomposed G into two forests.