Problem Set 2

Graph Theory 2016 – EPFL – Frank de Zeeuw & Claudiu Valculescu

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You can hand in one of the star problems before 10:15am on Thursday March 10th.

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1. Prove that the following statements about a graph G are equivalent.
   - G is a tree;
   - G is minimally connected (it is connected and removing any edge disconnects it);
   - G is maximally acyclic (it has no cycles and adding any edge creates a cycle).
2. Show that every tree T has at least $\Delta(T)$ leaves (vertices of degree 1).
3. Let T be a tree with $|V(T)| \geq 2$. Suppose the vertices of T are colored red and blue, such that no two adjacent vertices have the same color. Show that if there are at least as many red vertices as blue vertices, then T has a red leaf.
4. Show that a graph G contains at least $|E(G)| - |V(G)| + 1$ cycles.
5. A vertex is *central* if its greatest distance from any other vertex is as small as possible. Show that a tree has either a single central vertex, or two adjacent central vertices.
6. Determine the diameter of the following graphs.
   - A *hypercube*, whose vertex set is $\{0,1\}^d$, with an edge between two vectors if they differ in exactly one entry;
   - The *Petersen graph*, whose vertices are the 2-element subsets of $\{1,2,3,4,5\}$, with an edge between two subsets if they are disjoint;
   - A *visibility graph*, whose vertex set is a finite set $X \subset \mathbb{R}^2$, with an edge between two points if they can see each other, i.e., if the line segment between them contains no other point of X.
7. Prove that the following Greedy Forest-Growing algorithm returns a minimum-weight spanning tree, given a graph G and $w : E(G) \to \mathbb{R}_+$.

> **Greedy Forest-Growing Algorithm**
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> (1) Start with the empty graph F, and set $S = E(G)$;
>
> (2) Find $e \in S$ with minimum $w(e)$; if $S = \emptyset$ go to (4);
>
> (3) Add e to F, unless that creates a cycle; remove e from S; go back to (2);
>
> (4) If $|E(F)| = |V(G)| - 1$, return F, else return "disconnected".

8. Design a "greedy removal" algorithm for minimum-weight spanning trees, which starts with the whole graph and removes heavy edges. Prove that it works.

*9. Let T be a tree with t edges and G a graph. Prove that if $|E(G)| \geq t \cdot |V(G)|$, then T is a subgraph of G.

*10. Let $X = \{1, \ldots, n\}$ and let $\mathcal{S}$ be a set of n subsets of X. Use trees to prove that X contains an element x such that adding it to the sets of $\mathcal{S}$ gives n distinct subsets. In other words, there is an $x \in X$ such that $|\{S \cup \{x\} : S \in \mathcal{S}\}| = n$.
