Problem Set 12 – Solutions (without the bonus problems)

Graph Theory 2016 – EPFL – Frank de Zeeuw & Claudiu Valculescu

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1. *Determine* $R(P_3, P_3)$ *and* $R(K_{1,3}, K_3)$.

   • $R(P_3, P_3) = 5$: Let K_5 be 2-colored. Pick a vertex x_0. One color occurs at least twice at x_0, say it is red, and x_0x_1, x_0x_2 are red. If any of the edges from x_1, x_2 to the other two vertices x_3, x_4 is red, then we get a red P_3. Otherwise, for instance the path $x_1x_3x_2x_4$ is blue.
   For the lower bound, color K_4 with one triangle red, and the other three edges blue.
   • $R(K_{1,3}, K_3) = 7$: Let K_7 be colored red and blue. Pick a vertex x_0. If it has three red edges, we have a red $K_{1,3}$. Otherwise, x_0 has at least four blue edges $x_0x_1, x_0x_2, x_0x_3, x_0x_4$. If any edge between x_1, x_2, x_3, x_4 is blue, we get a blue K_3. Otherwise, x_1, x_2, x_3, x_4 form a red K_4, which contains a red $K_{1,3}$.
   To prove $R(K_{1,3}, K_3) > 6$, color K_6 so that the red edges form two red disjoint red triangles (so no red $K_{1,3}$), and the remaining edges are blue, which means they form a $K_{3,3}$ (so no blue K_3).

2. *Show that any* 2*-coloring of the edges of* K_6 *contains at least two monochromatic triangles.*

   There is one monochromatic triangle, say $x_1x_2x_3$ is red. Let x_4, x_5, x_6 be the other vertices. If the triangle $x_4x_5x_6$ is red, we are done, so we can assume that x_4x_5 is blue. If among the three edges from x_4 to x_1, x_2, x_3 there are two red edges, we would get a second red triangle, so we can assume that there are two blue edges from x_4 to x_1, x_2, x_3. Similarly, we can assume there are two blue edges from x_5 to x_1, x_2, x_3. Then there must be a blue edge from x_4 and a blue edge from x_5 that go the same vertex among x_1, x_2, x_3. This gives a blue triangle.

3. *Show that there exists an* N *such that if the integer box* $\{(x, y) : 1 \leq x, y \leq N\}$ *is* 2*-colored, then there is a monochromatic rectangle, i.e. a rectangle with all four corners the same color.*

   We claim that $N = 5$ works. Suppose that we have a coloring of the 5×5 box without a monochromatic rectangle. There must be points $(x_1, 1)$, $(x_2, 1)$, and $(x_3, 1)$ that all have the same colour, say red.

   For each i, at most one other point of the form (x_i, y) can be red. Indeed, suppose that for instance (x_1, y_1) and (x_1, y_2) are red, with $y_1, y_2 > 1$. Then each of the points $(x_2, y_1), (x_2, y_2), (x_3, y_1), (x_3, y_2)$ must be blue, or we'd get a red rectangle. But then these form a blue rectangle.

   Therefore, each of $(x_1, 1), (x_2, 1), (x_3, 1)$ has at least three blue points above it. Then there are two blue points above $(x_1, 1)$ which are at the same height as two blue points above $(x_2, 1)$, so they form a blue rectangle.

   For $N = 4$ it is not hard to construct a coloring without a monochromatic rectangle.

4. *Prove that every* 2*-coloring of* K_n *contains a monochromatic spanning tree.*

   Pick a vertex x. If all edges of x are red, they form a red spanning tree. Otherwise, x has a red edge xy and a blue edge xz. Apply induction to find a monochromatic spanning tree in $G - x$. If it is red, add xy, and if it is blue, add xz.

5. *Let T be a tree with t vertices. Prove that $R(K_s, T) = (s-1)(t-1)+1$.*

   Color $K_{(s-1)(t-1)+1}$ with red and blue. If every vertex has blue degree at least $t-1$, then by a lemma from Lecture 10, there is a blue T. Otherwise, there is a vertex x with blue degree at most $t-2$, so red degree at least $(s-1)(t-1)+1-(t-2) > (s-2)(t-1)+1$. Let H be the 2-colored complete subgraph on the at least $(s-2)(t-1)+1$ vertices that are connected to x by a red edge. By induction, H has a red K_{s-1} or a blue T. The red K_{s-1} would give a red K_s in the whole graph.

*6. *Determine $R(K_3, K_{2,2})$ and $R(K_{2,2}, K_{2,2})$.*

*7. *Let $2K_3$ be the graph consisting of two disjoint triangles. Prove that $R(2K_3, 2K_3) = 10$.*
