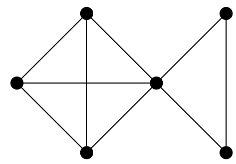
Problem Set 1 – Solutions

Graph Theory 2016 – EPFL – Frank de Zeeuw & Claudiu Valculescu

1. *For some n, give a graph with n vertices, $n+3$ edges, and exactly 8 cycles.*

   This is just a puzzle that you can solve by trial and error. Here is one possibility (the smallest one):

2. *Find two non-isomorphic graphs with the same number of vertices and the same sequence of degrees.*

   Let G_1 be of a cycle on 6 vertices, and let G_2 be the union of two disjoint cycles on 3 vertices each. In both graphs each vertex has degree 2, but the graphs are not isomorphic, since one is connected and the other is not.

   As another example, take the following two graphs.

   In both cases the degree sequence is $(3,3,2,1,1,1,1)$, but the graphs are not isomorphic, for instance because on the left the vertices of degree 3 are not adjacent, while on the right they are.

3. *Show that any graph with at least two vertices has two vertices with the same degree.*

   Suppose that the n vertices all have different degrees, and look at the set of degrees. Since the degree of a vertex is at most $n-1$, the set of degrees must be

   $$\{0, 1, 2, \ldots, n-2, n-1\}.$$

   But that's not possible, because the vertex with degree $n-1$ would have to be adjacent to all other vertices, whereas the one with degree 0 is not adjacent to any vertex.

4. *What is the maximum number of edges in a bipartite graph on n vertices? Prove it.*

   Let $G = (A \cup B, E)$ be a bipartite graph, with A, B disjoint and $|A|+|B| = n$. Since all the edges of G have one endpoint in A and the other in B, the number of edges $|E|$ of G cannot exceed the number of pairs $(a,b) \in A \times B$, so $|E| \leq |A| \cdot |B| = |A|(n-|A|)$. Intuitively, such a product is maximized when the two factors are equal, so when $|A| = \lfloor n/2 \rfloor$. More formally, we can use the inequality $4xy \leq (x+y)^2$ to get

   $$|E| \leq |A|(n-|A|) \leq \frac{(|A|+n-|A|)^2}{4} = \frac{n^2}{4}.$$

   Therefore, the number of edges of a bipartite graph on n edges is at most $n^2/4$.

   Note that $n^2/4$ is exactly the maximum when n is even, because then it is attained by the complete bipartite graph $K_{n/2,n/2}$. When n is odd, the maximum is actually $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil = \frac{n^2-1}{4}$, which is attained by $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

5. *Let G be a graph in which every vertex has even degree. Show that $E(G)$ can be partitioned into the edge sets of cycles.*

   Consider the following procedure. Start at a vertex v_1 of the graph. Pick an edge incident to v_1 and go along the edge to the neighboring vertex v_2. Pick an edge incident to v_2 (other than v_1v_2) and go along it to the vertex v_3. Continue building a path until we reach a vertex that we have already visited. Note that we can indeed repeat this procedure at every vertex except v_1, because the degrees are even (so if we reach a vertex we can get out again); the only vertex we can get stuck at is v_1, because we visited it at the start, after which it has an odd number of unvisited edges.

   When the procedure terminates, we have closed a cycle in the graph (if after v_k we revisit an earlier vertex v_ℓ, then $v_kv_{k+1}\cdots v_\ell v_k$ is a cycle). We remove this cycle. Note that the degree of each vertex remains even after removing the cycle, and the number of edges in the graph decreases. Repeating this whole procedure, successively eliminating cycles, we will eventually end up with an empty graph. Therefore, the edge set of the original graph was the disjoint union of the removed cycles.

6. *A graph is k-*regular *if every vertex has degree k. Describe all 1-regular graphs and all 2-regular graphs. Show that a graph on an odd number of vertices cannot be 3-regular, while for every even $n \geq 4$, there is a 3-regular graph on n vertices.*

   A 1-regular graph is just a disjoint union of edges (soon to be called a matching). A 2-regular graph is a disjoint union of cycles (this basically follows from Problem 5).

   Suppose there is a 3-regular graph G on $2k+1$ vertices. Since $2|E(G)| = \sum_{v\in V(G)} d(v)$, we would have $2|E(G)| = 3(2k+1) = 6k+3$, which is a contradiction.

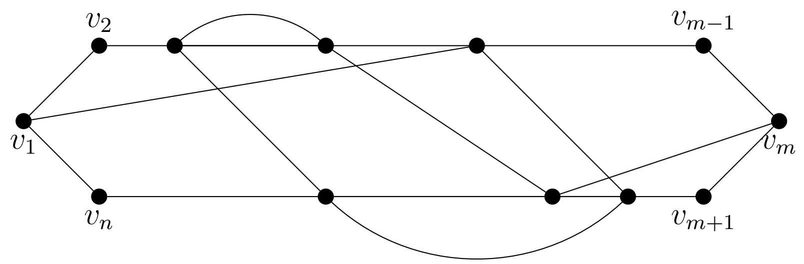
   For $n = 4$, K_4 is the only 3-regular graph with 4 vertices. We now construct a 3-regular graph on $2m$ vertices, for each $m \geq 3$. We take two cycles $u_1u_2\cdots u_m$, and $v_1v_2\cdots v_m$ of the same length, and we place an edge between u_i and v_i for each $1 \leq i \leq m$. The resulting graph is indeed 3-regular, since u_i is adjacent to u_{i-1}, u_{i+1}, and v_i, and similarly for v_i.

7. *Show that any graph with at least 6 vertices contains 3 vertices that are pairwise adjacent, or 3 vertices that are pairwise non-adjacent.*

   Pick an arbitrary vertex v_1. There are at least 5 other vertices, so by the pigeonhole principle, either v_1 has at least 3 neighbors, or it has at least 3 non-neighbors. In the first case, label the neighbors v_2, v_3, v_4. If any 2 of v_2, v_3, v_4 are adjacent, then with v_1 they form a pairwise adjacent triple; otherwise, no 2 are adjacent, so v_2, v_3, v_4 form a pairwise non-adjacent triple. Similarly, in the case where v_1 has 3 non-neighbors v_2, v_3, v_4, we get a pairwise non-adjacent triple if any 2 of v_2, v_3, v_4 are not adjacent, and otherwisev_2, v_3, v_4 are pairwise adjacent.

8. *Let G be a graph containing a cycle C, and assume that G contains a path of length at least k between two vertices of C. Show that G contains a cycle of length at least $\sqrt{k}$.*

The condition is a bit cryptic; another way of putting it is that there are two vertices v_1, v_m, with two disjoint paths between them, and a third path of length k. If the third path was disjoint from the other two, we would obviously have a cycle of length $\geq k$, but when the paths intersect there is trouble. We can picture the resulting subgraph like this (leaving out degree-2 vertices along the paths, so edges here are actually longer paths):



Note that one possibility not reflected in this picture is that the middle path might have edges in common with one of the other paths.

Let $C = v_1 v_2 \cdots v_n v_1$ be the original cycle, so its length is n. We can partition the middle path into "chords" $H_1, \ldots, H_p$, where H_i is a path $v_i \cdots v_j$ that only intersects C in v_i and v_j. Let ℓ_i be the length of H_i. If the middle path has an edge in common with one of the other paths, we simply take that edge as an H_i (technically not a chord, but it has the same properties that we need in the rest of the proof).

Observe that each chord H_i, combined with part of C, gives a cycle of length $\geq \ell_i$. Also, we have $p \leq n$, since each chord has a unique first point on C (in an arbitrary direction along the path from v_1 to v_m).

We can assume that $n < \sqrt{k}$, since otherwise we'd be done. Then suppose that $\ell_i < \sqrt{k}$ for all i. That would mean that

$$k = \sum \ell_i \leq p \cdot \sqrt{k} \leq n \cdot \sqrt{k} < \sqrt{k} \cdot \sqrt{k} = k,$$

which is a contradiction. Therefore some $\ell_i \geq \sqrt{k}$, so that H_i together with part of C gives a cycle of length $\geq \sqrt{k}$.

*9. *Prove the statement of Problem 8 with $\sqrt{2k}$ instead of $\sqrt{k}$.*

First of all, each chord H_i cuts C in two, and the larger part has $\lceil n/2 \rceil \geq n/2$ edges, so in fact we get a cycle with $\geq \ell_i + n/2$ edges.

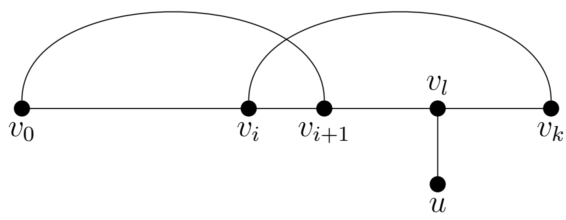
The average length of a chord H_i is $\frac{k}{p} > \frac{k}{n}$. Since they can't all be smaller than the average, some H_i must have length $\ell_i \geq \frac{k}{n}$. Then the cycle we get from it has length $\geq \frac{k}{n} + \frac{n}{2}$. Let's optimize the function $f(x) = \frac{k}{x} + \frac{x}{2}$ using calculus: $f'(x) = \frac{-k}{x^2} + \frac{1}{2}$, so the only positive extremum is $x = \sqrt{2k}$, which is a minimum. So of all the n, we get the smallest cycle for $n = \sqrt{2k}$, and then the length of that cycle is still $\geq \frac{k}{\sqrt{2k}} + \frac{\sqrt{2k}}{2} = \sqrt{2k}$. Hence we are guaranteed to have a cycle of length $\sqrt{2k}$.

*10. *Show that every connected graph G contains a path of length at least* $\min\{2\delta(G), |G|-1\}$.

Take a longest path $P = v_0 v_1 \cdots v_k$. If $k = |G| - 1$, then we are done, so we can assume that there is a vertex not on P. Then by connectedness there must be a vertex u that is not on P but adjacent to a vertex of P, let's say v_l.

Observe that v_l is not v_0 or v_k, since then we could extend P to a longer path. Also, v_0 and v_k are not adjacent, otherwise $uv_l \cdots v_k v_0 \cdots v_{l-1}$ would be a longer path. And the neighbourhoods $N(v_0)$ and $N(v_k)$ are contained in $\{v_1, \ldots, v_k\}$, again since otherwise we could the extend P.

One more observation, which is a bit trickier: we cannot have anything like (not showing all the vertices on P)



since this would also give a longer path:

$$v_{l+1} \cdots v_k v_i \cdots v_0 v_{i+1} \cdots v_l u.$$

Note that something similar works when $i > l$. Thus we cannot have v_i adjacent to v_k and at the same time v_{i+1} adjacent to v_0.

The set $\{v_1, \ldots, v_k\}$ contains the two sets $N(v_0)$ and $\{v_{i+1} : v_i \in N(v_k)\}$, both of which have size at least $\delta(G)$. Our last observation implies that these two sets are disjoint, which tells us that $k \geq 2\delta(G)$.