Practice exam – Solutions

Graph Theory 2016 – EPFL – Frank de Zeeuw & Claudiu Valculescu

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1. *Show that a graph is bipartite if and only if it has no odd cycles.* `[From the notes]`

   In a bipartite graph, any path that starts in one part and ends in the other must have an odd number of edges. Any cycle consists of such a path with another edge added, so it must have even length.

   Suppose a graph has no odd cycles. We can assume that it is connected, since if every connected component is bipartite, then so is the graph itself. Take a spanning tree of the graph, which we can think of as having a root and several levels (the first level consists of all vertices adjacent to the root, the second level consists of all remaining vertices adjacent to the second level, etc.). Place the vertices of the even levels (including the root) in the set A, and the vertices of the odd levels in the set B. For two vertices in the same part, the unique path between them in the tree has even length, because to go from one even level to the next is must pass through one odd level, and vice versa. Therefore, since there are no odd cycles, vertices in the same part are not adjacent, because that would make the even path into an odd cycle. Hence $A \cup B$ is a bipartition of the graph.

2. *Suppose that G is a graph with $|V(G)| \geq 3$ such that for all $x \in V(G)$, $G - x$ is a tree. Determine what kind of graph G is, and prove it.* `[New problem]`

   It is a cycle. First of all, it contains a cycle, since if it was a tree with at least three vertices, there would be a vertex whose removal would disconnect the graph. Moreover, any cycle must contain every vertex, because otherwise removing the vertex would leave the cycle intact. Thus the graph has a Hamilton cycle C.

   Suppose there is an edge $xy \notin E(C)$. This would create two cycles that share only xy, and removing a vertex other than x or y would leave the other cycle intact. Thus the given graph is equal to the cycle C.

3. *Show that a matching $M \subset E(G)$ is a maximum matching if and only if there is no augmenting path for M.* `[From the notes]`

   We prove that M is not maximum if and only if there is an augmenting path for M. First suppose that there is an augmenting path Q for the matching M. Replace M by $M' = M \triangle E(Q)$. Then M' is a matching, since the endpoints of Q were unmatched by M, and we have $|M'| > |M|$, so M is not maximum.

   Suppose M is not maximum, so there is a matching M' with $|M'| > |M|$. Let D be the graph with $V(D) = V(G)$ and $E(D) = M \triangle M'$. Every vertex has degree 0, 1, or 2 in D. This implies that the connected components of D are either cycles, paths, or isolated vertices. A cycle in D has the same number of edge from M as from M', so $|M'| > |M|$ implies that there is a path P in D with more edges from M' than from M. Then P must be an augmenting path for M.

4. *Show that if every two odd cycles of G intersect in at least one vertex, then $\chi(G) \leq 5$.* `[From the problem sets]`

   If G has no odd cycles, then it is bipartite (by question 1), which means $\chi(G) = 2$. Let C be any odd cycle, and remove its vertices from G to get a new graph $G - C$. It has no odd cycles, since every odd cycle previously intersected C. This implies that $G - C$ is bipartite, or in other words 2-colorable. Then we can combine a 2-coloring of $G - C$ with a 3-coloring of C to get a 5-coloring of G.

5. *Let G be a graph containing a cycle. Prove that $\alpha(G) \geq \left\lfloor \frac{1}{2}\mathrm{gir}(G) \right\rfloor$.* `[New problem]`

   Let $C = x_1 x_2 \cdots x_k x_1$ be a shortest cycle, so that its length equals $k = \mathrm{gir}(G)$. Let I be the set of alternating vertices $x_2, x_4, \ldots$, with the last vertex being x_{k-1} if k is odd and x_k if k is even. Then I is an independent set. Indeed, if there was an edge between two vertices in I, replacing part of the cycle by that edge gives a shorter cycle. Since $|I| = \lfloor \frac{1}{2}k \rfloor$, we have $\alpha(G) \geq \lfloor \frac{1}{2}k \rfloor = \lfloor \frac{1}{2}\mathrm{gir}(G) \rfloor$.

6. *Prove that a graph is 2-connected if and only if for every three vertices x, y, z, there is a path from x to z that passes through y.* `[From the problem sets]`

   if: Consider $G - z$ for $z \in V(G)$, and take any $x, y \in V(G - z)$. The condition gives that there is a path P in G from x to z through y. This path has a subpath from x to y, which does not pass through z, since otherwise P would not be a path. Thus there is a path from x to y in $G - z$, which means that $G - z$ is connected. The fact that this is true for every $z \in V(G)$ means that G is 2-connected.

   only if: Assume G is 2-connected. Add a vertex w that is connected to both x and z; the resulting graph is still 2-connected. By Whitney's Theorem, there are two internally disjoint paths from w to y. Since x and z are the only neighbors of w, one of the paths must pass through x and the other must pass through z. Remove w and take the path from x to y, followed by the path from y to z. Since the two paths were internally disjoint, this really is a path.

7. *State and prove Euler's formula for connected planar graphs.* `[From the notes]`

   Theorem (Euler)**.** *Let G be a connected planar graph and D a planar drawing of G. Then $|V(G)| - |E(G)| + |F_D(G)| = 2$.*

   Proof. We use induction on $|E(G)|$. Since G is connected, we have $|E(G)| \geq |V(G)| - 1$, so we can start with the base case $|E(G)| = |V(G)| - 1$. In that case, G is a tree, so any planar drawing has only one face and we have $|F_D(G)| = 1$. Then

$$|V(G)| - |E(G)| + |F_D(G)| = |V(G)| - (|V(G)| - 1) + 1 = 2.$$

   Assume that $|E(G)| > |V(G)| - 1$. Then G contains a cycle C. Pick an edge $e \in E(C)$. The graph $G - e$ is connected, and the drawing D directly gives a planar drawing D' of $G - e$. Moreover, since e is in a cycle, it is not a cut-edge of G, so e is on the boundary of two different faces (one inside C, one outside C; here we use the Jordan Curve Theorem). Removing e merges these two faces, and does not affect any other faces, so $|F_{D'}(G - e)| = |F_D(G)| - 1$. By induction, we have

$$|V(G - e)| - |E(G - e)| + |F_{D'}(G - e)| = 2.$$

   Plugging in $|V(G-e)| = |V(G)|$, $|E(G-e)| = |E(G)| - 1$, and $|F_{D'}(G-e)| = |F_D(G)| - 1$ gives Euler's formula for G. $\square$

8. *Prove that if G is a graph with $|V(G)| \geq 4$ and $|E(G)| \geq 2|V(G)| - 3$, then G contains a cycle with a chord (a* chord *of a cycle is an edge that is not part of the cycle, but that connects two vertices from the cycle).* `[New problem]`

   If $\delta(G) < 3$, then there is a vertex x with $d(x) \leq 2$. Removing x, we have

$$|E(G-x)| \geq |E(G)| - 2 \geq (2|V(G)| - 3) - 2 = 2(|V(G)| - 1) - 3 = 2|V(G-x)| - 3,$$

   so by induction $G - x$ contains a cycle with a chord, hence so does G. The base case of the induction is $|V(G)| = 4$. In that case $|E(G)| \geq 5$, so G is a C_4 with a chord.

   Assume $\delta(G) \geq 3$. Take a longest path $P = x_1 \cdots x_k$. As usual, x_1 cannot have neighbors outside the path, since otherwise we could extend the path. Since $d(x_1) \geq \delta(G) \geq 3$, x_1 has at least two neighbors x_i, x_j on the path besides x_2, with $2 < i < j$. Then $x_1 \cdots x_j x_1$ is a cycle with the chord $x_1 x_i$.

9. *Prove that a graph with n vertices and e edges has at least $\frac{e}{n}\left(e - \frac{1}{4}n^2\right)$ triangles.* `[From the problem sets]`

   The number of triangles based on an edge xy is

$$|N(x) \cap N(y)| = |N(x)| + |N(y)| - |N(x) \cup N(y)| \geq d(x) + d(y) - n.$$

   Hence the total number of triangles is at least

$$\frac{1}{3} \sum_{xy \in E(G)} (d(x) + d(y) - n) = \frac{1}{3}\left(\sum_{x \in V(G)} d(x)^2\right) - \frac{en}{3} \geq \frac{1}{3n}(2e)^2 - \frac{en}{3} = \frac{4e}{3n}\left(e - \frac{n^2}{4}\right).$$

   Obviously $\frac{4}{3} > 1$. In the inequality we used Cauchy-Schwarz.