Planar graphs

Lecture 9 – Graph Theory 2016 – EPFL – Frank de Zeeuw

1 Drawings of graphs

We have so far only considered *abstract graphs*, although we often used pictures to illustrate the graph. In this lecture, we prove some facts about pictures of graphs and their properties. To set this on a firm footing, we give a formal definition of what we mean by "picture", although in most of the lecture we will be less formal.

Definition. *A* drawing *of a graph G consists of an injective map $f : V(G) \to \mathbb{R}^2$, and a curve C_{xy} (the image of an injective continuous map $[0,1] \to \mathbb{R}^2$) from x to y for every $xy \in E(G)$, such that if $z \neq x, y$ then $z \notin C_{xy}$. A drawing is* planar *if the curves C_{xy} do not intersect each other except possibly at endpoints.*

Definition. *A graph is* planar *if it has a planar drawing.*

For example, the graphs K_4 and $K_{2,3}$ are planar graphs.

Warning: Some proofs in this lecture will be not quite rigorous. This is mainly because arbitrary continuous curves can behave in unintuitive ways, which makes it hard to make intuitive arguments fully rigorous. Whenever an argument is not quite rigorous, we will point out why. In most cases, the arguments can be made rigorous using the *Jordan Curve Theorem*, which states that a non-self-intersecting closed curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into an *inside* and an *outside*, and any path between a point inside and a point outside must pass through the closed curve. Although this statement is very intuitive, proving it turns out to be quite hard. We will not try to prove it in this course, since it would take us too far into topology and away from graph theory.

Definition. *A* face *of a drawing D of a graph G is a maximal connected set in $\mathbb{R}^2$ after the vertices and edges of D are removed. We write $F_D(G)$ for the set of faces of D.*

Every drawing has an *outer face*, which is the unique unbounded face in the complement of the drawing. Note that although typically an edge lies in the boundary of two faces, an edge can be in the boundary of a single face, which happens if and only if the edge is a cut-edge; this fact will create some complications later on.

2 Non-planar graph

To show that a graph is planar, we only have to supply a planar drawing. It is often a little harder to show that a graph is not planar.

Proposition 2.1. *The graph K_5 is not planar.*

Proof. The graph contains a K_3, which can basically be drawn in only one way. If in a drawing the fourth vertex is inside this K_3 and the fifth is outside, then the edge between them must cross the K_3, which means the drawing is not planar. If both vertices are inside the K_3, then the three edges of one vertex divide the inside face into three faces. The other vertex must then be in one of these three faces, and one of its edges must cross the boundary of this face. A similar argument applies when both vertices are outside the K_3.

Note that to make this proof rigorous, we would need the Jordan Curve Theorem to show that a drawing of a K_3 has an inside and an outside, and that to connect a vertex inside to a vertex outside we would have to cross the K_3. $\square$

Proposition 2.2. *The graph $K_{3,3}$ is not planar.*

Proof. The graph contains a C_6, and the remaining three edges connect opposite corners of the C_6 in a symmetric way. Two of these three edges would have to be both inside or outside the C_6, and there one of them would create a closed curve that the other one would have to cross. Again, this proof relies on the Jordan Curve Theorem. $\square$

If a graph H is a subgraph of a graph G and H is not planar, then G is also not planar, since a planar drawing of G would give a planar drawing of H. A stronger version of this is the following. Call a graph H' a *subdivision* of H if it can be obtained from H by repeatedly replacing an edge xy by a path xzy for some new vertex z (informally, we simply place the new vertex z somewhere on top of the edge xy, subdividing it into two edges xz and zy). It is easy to see that a subdividion of H is planar if and only if H is planar; the new vertices do not have any effect on whether ot not we can draw the graph without crossing edges.

Proposition 2.3. *The Petersen graph is not planar.*

Proof. It contains a subdivision of $K_{3,3}$. $\square$

Amazingly, any non-planar graph can be shown to be non-planar by displaying a subdivision of K_5 or $K_{3,3}$ in that graph, and a planar graph can be shown to be planar by showing that it does not contain such a subdivision. This is the content of the following theorem. Note that it states an equivalence between a topological statement (the graph having a planar drawing) and a combinatorial statement (the graph not containing a certain subdivision). The proof of this theorem is fairly hard and we will not include it in this course.

Theorem 2.4 (Kuratowski)**.** *A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.*

3 Euler's formula

Theorem 3.1 (Euler)**.** *Let G be a connected planar graph and D a planar drawing of G. Then*

$$|V(G)| - |E(G)| + |F_D(G)| = 2.$$

Proof. We use induction on $|E(G)|$. Since G is connected, we have $|E(G)| \geq |V(G)| - 1$, so we can start with the base case $|E(G)| = |V(G)| - 1$. In that case, G is a tree, so any planar drawing has only one face and we have $|F_D(G)| = 1$. Then

$$|V(G)| - |E(G)| + |F_D(G)| = |V(G)| - (|V(G)| - 1) + 1 = 2.$$

Assume that $|E(G)| > |V(G)| - 1$. Then G contains a cycle C. Pick an edge $e \in E(C)$. The graph $G - e$ is connected, and the drawing D directly gives a planar drawing D' of $G - e$. Moreover, since e is in a cycle, it is not a cut-edge of G, so e is on the boundary of two different faces (one inside C, one outside C; here we use the Jordan Curve Theorem). Removing e

merges these two faces, and does not affect any other faces, so $|F_{D'}(G-e)| = |F_D(G)| - 1$. By induction, we have

$$|V(G-e)| - |E(G-e)| + |F_{D'}(G-e)| = 2.$$

Plugging in $|V(G-e)| = |V(G)|$, $|E(G-e)| = |E(G)| - 1$, and $|F_{D'}(G-e)| = |F_D(G)| - 1$ gives Euler's formula for G. □

Note that Euler's formula implies that for two planar drawings D_1, D_2 of a graph we have $|F_{D_1}(G)| = 2 - |V(G)| + |E(G)| = |F_{D_2}(G)|$, so the number of faces in a planar drawing of a graph is independent of the drawing.

Proposition 3.2. *Let G be a planar graph with $|V(G)| \geq 3$. Then $|E(G)| \leq 3|V(G)| - 6$.*

Proof. We can assume that G is *maximally planar*, i.e., that adding any edge to G would make it non-planar. Indeed, given any planar graph, we can add edges until it is maximally planar; if the bound holds for the maximally planar graph, then it also holds for the original graph. Then G is connected, every edge is in the boundary of two faces, and every face has exactly three edges on its boundary, since if any of these properties did not hold, then we could add an edge.

Double counting the pairs (e, f), where the edge e is on the boundary of the face f, gives us

$$2|E(G)| = 3|F_D(G)|.$$

Plugging $|F_D(G)| = \frac{2}{3}|E(G)|$ into Euler's formula gives

$$2 = |V(G)| - |E(G)| + \frac{2}{3}|E(G)|,$$

which rearranges to $|E(G)| = 3|V(G)| - 6$. Since this equality holds for any maximally planar graph, the inequality $|E(G)| \leq 3|V(G)| - 6$ holds for any planar graph. □

It is possible to prove Proposition 3.2 in a more direct way, without assuming the graph to be maximally planar. Basically, every edge bounds at most two faces, and every face is bounded by at least three edges, so we should have $2|E(G)| \geq 3|F_D(G)|$. Plugging that into Euler's formula directly gives the inequality $|E(G)| \leq 3|V(G)| - 6$ (to apply Euler's formula, we need the graph to be connected, but we can apply it to each connected component, which leads to the same bound). However, it is not always true that every edge bounds at most two faces and every face is bounded by at least three edges; take for instance the path P_2, which has one face bounded by two edges. For P_2 we have $2|E(G)| \geq 3|F_D(G)|$ anyway, but it is tricky to prove that this holds for all graphs. The step of making the graph maximally planar is a convenient way around this.

Proposition 3.2 gives us a quick proof that K_5 is not planar:

$$|E(K_5)| = \binom{5}{2} = 10 > 9 = 3 \cdot 5 - 6 = 3|V(K_5)| - 6.$$

This does not quite work for $K_{3,3}$ or the Petersen graph, but their non-planarity can also be proved using Euler's formula (see Problem Set 9 for $K_{3,3}$).

Corollary 3.3. *If G is planar, then it has a vertex of degree at most five.*

Proof. Combining Proposition 3.2 with the degree formula from the first lecture gives

$$\sum_{v \in V(G)} d(v) = 2|E(G)| \leq 6|V(G)| - 12.$$

This implies that the average degree of a vertex is strictly less than six, so there must be a vertex with degree at most five. □

4 Coloring planar graphs

Theorem 4.1 (Five Color Theorem)**.** *If G is planar then $\chi(G) \leq 5$.*

Proof. Fix a planar drawing of G. We use induction on $|V(G)|$. The statement is obvious for $|V(G)| \leq 5$. By Corollary 3.3, there is a vertex $v \in V(G)$ of degree at most five. By induction, we can color $G - v$ with five colors. If this coloring uses at most four colors on $N(v)$, then we can color v with the fifth color and we are done. Thus we can assume that v has five neighbors $x_1, \ldots, x_5$, and that x_i has color i. We can also assume that the edges $vx_1, \ldots, vx_5$ leave v in that order when we go around v in the clockwise direction (say).

We call a path in G an ij-path if all its vertices have color i or j.

Suppose that there is no 13-path from x_1 to x_3. Let R be the set of vertices that are reachable from x_1 by a 13-path. By assumption, x_3 is not in R. Then we can swap the colors 1 and 3 for all the vertices in R. This gives a valid coloring, and it leaves x_1 and x_3 both colored with 3. Then we can color v with 1 and we are done.

Now suppose that there is a 13-path from x_1 to x_3; together with v this path forms a cycle C, all of whose vertices are colored with 1 or 3 (or uncolored in the case of v). Let S be the set of vertices reachable by 24-paths from x_2. Then the cycle C separates S from x_4 (here we use the Jordan Curve Theorem), so x_4 is not in S. Thus we can swap colors 2 and 4 in S, and then color v with 2. □

This theorem is actually not best possible. The famous Four Color Theorem states that for any planar graph we have $\chi(G) \leq 4$. However, the proof is extremely hard and has only been completed using computers to check many cases.