k-connected graphs

Lecture 8 – Graph Theory 2016 – EPFL – Frank de Zeeuw

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1 Menger's Theorem

In this lecture we generalize our results from the previous lecture, from 2-connected graphs to k-connected graphs. We first prove the key tool behind these generalizations, which is *Menger's Theorem*. Its proof is relatively difficult and requires a few ad hoc definitions.

Given two vertex sets $A, B \subset V(G)$, we call a path an *AB-path* if one endpoint is in A, the other endpoint is in B, and no other vertex on the path is in A or B. If $x \in A \cap B$, then we consider x itself to be an AB-path. We call a set $S \subset V(G)$ such that every AB-path intersects S an *AB-separator*.

Theorem 1.1 (Menger)**.** *Let G be a graph and $A, B \subset V(G)$. The maximum number of disjoint AB-paths equals the minimum size of a set $S \subset V(G)$ such that every AB-path contains a vertex from S.*

Proof. Let s be the minimum size of an AB-separator. Any set of disjoint AB-paths contains at most s paths, since otherwise a separator of size s could not cut all the paths. We have to show that there exists a set of s disjoint AB-paths. We will prove this using induction on $|E(G)|$. When $|E(G)| = 0$, then the only minimum AB-separator is $A \cap B$ itself, so we must have $s = |A \cap B|$, and the s vertices in $A \cap B$ are disjoint AB-paths.

Assume $|E(G)| \geq 1$ and let xy be any edge in $E(G)$. We claim that if G does not have s disjoint AB-paths, then G has an AB-separator S of size s that contains both x and y. We first show how to prove the theorem given this claim, and then we will prove the claim.

Consider the graph $G - xy$, in which S is still an AB-separator. Consider an AS-separator R in $G - xy$. Observe that every AB-path in G contains an AS-path in $G - xy$ (this could fail if we did not have $x, y \in S$), so R is also an AB-separator in G, which implies $|R| \geq s$. It follows that the minimum size of an AS-separator in $G - xy$ is at least s. By induction, this implies that $G - xy$ has s disjoint AS-paths. By the same argument on the other side, $G - xy$ has s disjoint SB-paths. These AS-paths and SB-paths do not intersect outside S, since that would give an AB-path avoiding S. Since $|S| = s$, we can connect the s AS-paths with the s SB-paths to obtain s disjoint AB-paths.

It remains to prove the claim that if G does not have s disjoint AB-paths, then G has an AB-separator S of size s containing x and y. We *contract* xy to get the graph G/xy; this means that we remove xy and identify x and y to a vertex v_{xy}. More precisely, we remove x, y, and all incident edges, add a new vertex v_{xy}, and we connect v_{xy} to every vertex that was previously adjacent to x or y (or both). We modify A in the natural way: If A does not contain x or y, then we let A' be the same set in G/xy; if A does contain x or y (or both), then we set $A' = (A \backslash \{x, y\}) \cup \{v_{xy}\}$. We do the same with B to get B'.

An $A'B'$-path P in G/xy corresponds to one or more AB-paths in G: If v_{xy} is not on P, then P is an AB-path in G; if v_{xy} is on P, then there can be more than one way to modify P to an AB-path in G. Conversely, an AB-path in G corresponds to an $A'B'$-path in G/xy. Also note that disjoint $A'B'$-paths in G/xy correspond to disjoint AB-paths in G.

By assumption, G does not have s disjoint AB-paths, so G/xy does not have s disjoint $A'B'$-paths. By induction (for the statement of the theorem, not just the claim), this implies that the minimum size of an $A'B'$-separator in G/xy is at most $s-1$, so G/xy has an $A'B'$-separator S' of size at most $s-1$. We must have $v_{xy} \in S'$, since otherwise S' would be an AB-separator in G that is smaller than the minimum (using the fact that every AB-path in G corresponds to an $A'B'$-path in G/xy). Set $S = (S' \backslash \{v_{xy}\}) \cup \{x, y\}$. It is an AB-separator in G of size s that contains x and y. □

Corollary 1.2. *Let G be a graph and $x, y \in V(G)$ distinct vertices with $xy \notin E(G)$. The maximum number of internally disjoint paths from x to y equals the minimum size of a set $S \subset V(G) \backslash \{x, y\}$ with the property that every path from x to y contains a vertex from S.*

Proof. Apply Theorem 1.1 with $A = N(x)$ and $B = N(y)$. Note that adding x and y to an AB-path gives a path from x to y, but removing x and y from a path between x and y may not give an AB-path, because a path between x and y may pass through more than one neighbor of x or y. Nevertheless, every path between x and y does contain a subpath that is an AB-path, so the maximum number of internally disjoint paths from x to y equals the number of disjoint AB-paths. By Theorem 1.1, this maximum equals the minimum size of a set $S \subset V(G)$ such that every AB-path intersects S. Any such set S also has the property that every path from x to y contains a vertex of S. It remains to show that a minimum such set S does not contain x or y. If S contains x (say) and every AB-path intersects S, then the same holds for $S \backslash \{x\}$, since a path that goes from B to A via x contains a subpath from B to A that does not contain x, and that subpath must contain some other vertex of S. □

2 k-connected graphs

Recall that for $S \subset V(G)$, $G - S$ is the subgraph obtained from G by removing the vertices of S and all edges incident with a vertex of S.

Definition. *A graph is* k-connected *if $|V(G)| > k$ and for every $S \subset V(G)$ with $|S| = k-1$ the graph $G - S$ is connected.*

Note that 1-connected is the same as connected, except (annoyingly) when $|V(G)| = 1$.

Theorem 2.1. *A graph G is k-connected if and only if for any distinct $x, y \in V(G)$ there are k internally disjoint paths from x to y in G.*

Proof. If G contains k internally disjoint paths between any two vertices, then we must have $|V(G)| > k$, and after removing $k-1$ vertices, any two vertices are still connected by one of the k paths, so G is k-connected.

Conversely, suppose that G is k-connected and contains two vertices x, y that are not connected by k internally disjoint paths. If $xy \notin E(G)$, then by Corollary 1.2 there are $k-1$ vertices whose removal disconnects x from y, contradicting k-connectedness. Thus we can assume that $xy \in E(G)$.

By assumption, any set of internally disjoint paths connecting x and y has size at most $k-1$. Moreover, any maximum such set includes the path xy, since otherwise we can add it. Thus $G - xy$ has at most $k-2$ internally disjoint paths from x to y. By Corollary 1.2, there is a set $S \subset V(G) \backslash \{x, y\}$ of size $k-2$ such that every path from x to y intersects S. We have $|S \cup \{x, y\}| = k$ and $|V(G)| > k$, so there is a vertex $z \notin S \cup \{x, y\}$. In $G - xy$, S must separate z from either x or y, since otherwise there would be a path from x to y avoiding S. Say that S separates z from x; then $S \cup \{y\}$ is a set of at most $k-1$ vertices that separates z from x in G, contradicting the k-connectedness of G. □

Definition. *The* connectivity $\mathrm{conn}(G)$ *of a graph G is the maximum k such that G is k-connected.*

We have $\mathrm{conn}(C_n) = 2$, $\mathrm{conn}(K_n) = n-1$ for $n \geq 3$, and the connectivity of the Petersen graph is 3. Note that $\mathrm{conn}(G) \leq \delta(G)$, since if x is a vertex with $d(x) = \delta(G)$, then $N(x)$ is a set of $\delta(G)$ vertices such that $G - N(x)$ is not connected.

Finally, we prove the generalization of the fact from the last lecture that in a 2-connected graph every two vertices are contained in a cycle. The proof of that fact was easy, since two internally disjoint paths between two vertices give a cycle containing those two vertices. For more than two vertices, the proof is more interesting. We require the following lemma.

Lemma 2.2. *Let G be a k-connected graph. For every $x \in V(G)$ and $U \subset V(G)$ with $|U| \geq k$, there are k paths from x to U that are disjoint aside from x, with each path having exactly one vertex from U.*

Proof. Add a vertex u that is adjacent to all the vertices in U; the resulting graph is still k-connected (this requires that $|U| \geq k$). By Theorem 2.1, there are k internally disjoint paths from x to u. Removing u from these paths gives paths from x to U that are disjoint aside from x. If such a path contains more than one vertex of U, then it contains a subpath with exactly one vertex from U, which we can take instead. $\square$

Theorem 2.3 (Dirac)**.** *If a graph G is k-connected for $k \geq 2$, then for every set K of k vertices in G, there exists a cycle in G containing K.*

Proof. We use induction on k. For $k = 2$, we saw a proof in the previous lecture; it also follows directly from the case $k = 2$ of Theorem 2.1. Assume $k > 2$ and pick any $x \in K$. By induction, G has a cycle C containing $K \backslash \{x\}$. If $x \in V(C)$, we are done, so we can assume that $x \notin V(C)$.

Suppose that $|V(C)| = k-1$. By Lemma 2.2 and the fact that G is $(k-1)$-connected, there are $k-1$ paths from x to C that are disjoint aside from x, each containing exactly one vertex of C. We can use any two of the paths from x that end at adjacent vertices $y, z \in V(C)$ to obtain a cycle containing x as well as $K \backslash \{x\}$: Remove the edge yz from C, and replace it by the path that goes from y to x and then from x to z. Since these paths were disjoint aside from x, and also contain no other vertices from C, this indeed gives a cycle.

Suppose that $|V(C)| \geq k$. By Lemma 2.2 and the fact that G is k-connected, there are k paths from x to C that are disjoint aside from x, each containing exactly one vertex of C. The $k-1$ vertices of $K \backslash \{x\}$ partition $V(C)$ into $k-1$ consecutive segments (for each vertex of $K \backslash \{x\}$, take that vertex and all the following ones, up to the one before the next vertex of $K \backslash \{x\}$). By the pigeonhole principle, there must be a segment that is reached by two of the paths from x. Then as in the previous case these two paths can be used to obtain a cycle containing x as well as $K \backslash \{x\}$. $\square$