Introduction to Combinatorics Spring, 2011

Homework 8 – Probabilistic Methods

Janos Pach & Nabil Mustafa *14 April*

Questions

1. Let $G = (V, E)$ be a graph with $2n$ vertices and m edges. Prove that there exists a cut with n vertices on each side, and with at least $\frac{mn}{2n-1}$ edges across the cut.

2. Given a graph $G = (V, E)$ with n vertices, consider the following method for constructing a cut of G. Let $V = \{v_1, \ldots, v_n\}$ be the vertices of G. Start with the cut consisting of vertices v_1 and v_2 on two different sides, i.e., the cut $(\{v_1\}, \{v_2\})$. Now add each of the remaining vertices $v_3, \ldots, v_n$ to this cut one-by-one as follows: assume we have already added the vertices $v_3, \ldots, v_{i-1}$ to the cut. Then add the next vertex v_i to the side of the cut to which v_i has *fewer* edges. Prove that the final cut has at least $m/2$ edges across it.

3. Consider the following deterministic way of constructing ϵ-nets: pick an element of X that hits maximum number of sets. Add this element to our ϵ-net, and inductively compute an ϵ-net for the remaining sets that were not hit. Show that this constructs an ϵ-net of size $2k \log m$.

4. Pick k random numbers from the set $\{1, \ldots, n\}$ (a number may be picked multiple times). Show that the expected value of the minimum number picked is approximately $\frac{n}{k+1}$. You may use the fact that $\sum_{i=1}^{n} i^m$ is approximately $n^{m+1}/(m+1)$.

Bonus Problem. Let $\mathcal{T}$ be the family of all nonempty subsets of $\{1, 2, \ldots, n\}$ with the property that any $T \in \mathcal{T}$ contains no two consecutive integers. For every $T \in \mathcal{T}$, let p_T denote the product of the squares of all elements of T. Prove that the sum of the numbers p_T over all elements $T \in \mathcal{T}$ is $(n+1)! - 1$.

10 points.