Introduction to Combinatorics Spring, 2011

Homework 7 – Probabilistic Methods

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Questions

1. We toss a fair coin n times. What is the expected number of 'runs'? Runs are consecutive tosses with the same result. For instance, the toss sequence HHHTTHTH has 5 runs.

2. For a permutation π, let $f(\pi)$ be the number of fixed points of π. What is $E[f(\pi)]$ for a random permutation π on n elements.

3. The number of *left maxima* for a permutation π of $\{1, \ldots, n\}$ is defined to be the number of indices $i \in [n]$ such that $\pi(i) > \pi(j)$ for all $j < i$. Using linearity of expectation, compute the expected number of left maxima for a random permutation?

4. Let X be a set of n elements, and $\mathcal{M}$ a set system on X, i.e., $\mathcal{M} = \{S_1, \ldots, S_m\}$, where $S_i \subseteq X$ and $|S_i| = k$ for all $i = 1 \ldots m$. Prove that if $m < 2^{k-1}$, then X can be two-colored (i.e., each element of X can be colored either 'red' or 'blue') such that no set S_i is monochromatic (a set S is monochromatic if all the elements in S have the same color).

5. Can you construct a tournament T on 6 vertices such that for *any* pair of vertices $u, v \in T$, there is a third vertex w such that w beats both u and v? What about a tournament with 7 vertices?

Bonus Problem. Prove that there exist four positive integers a_1, a_2, a_3, a_4 such that for any integer $w \in \{1, \ldots, 40\}$, there exist $c_i \in \{-1, 0, +1\}$, $i = 1, \ldots, 4$, such that $w = c_1 \cdot a_1 + c_2 \cdot a_2 + c_3 \cdot a_3 + c_4 \cdot a_4$.

10 points.