

Homework 6 – DILWORTH'S THEOREM

*Janos Pach & Nabil Mustafa**31 March***Questions**

1. Prove the following: Let k, l be natural numbers. Then every sequence of real numbers of length $kl + 1$ contains a non-decreasing subsequence of length $k + 1$ or a decreasing subsequence of length $l + 1$.
2. Let S be a sequence of n (not necessarily distinct) integers. Assume $n > rst$, where r, s, t are three positive integers. Then prove that either there exists a strictly increasing subsequence of size r , or a strictly decreasing subsequence of size s or a subsequence of size t consisting of the same integer.
3. Given a set I of n intervals in \mathbb{R} , assume that there is no 'nested' set of intervals with size k (a set of intervals are nested if for every pair, one is completely contained inside the other). Then prove that there exists a subset of size n/k where no pair of intervals are nested.
4. Given a set I of n intervals in \mathbb{R} , prove that either one can find \sqrt{n} disjoint intervals in I , or \sqrt{n} intervals $I' \subseteq I$ where all the intervals in I' contain a common point.

Bonus Problem. Alice and Bob play a game where they have to write bits, which can be either 0 or 1, one after the other on a piece of paper. Alice will write the first bit, Bob writes the next bit, then Alice, then Bob and so on. The player who writes the bit after which there are two repeated sequences (can be disjoint, or overlapping) of length n loses. Show that the game always ends, and that for odd n , Bob always wins.

10 points.