

Homework 4 – DOUBLE-COUNTING ARGUMENTS

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Questions

1. You are given a set P of n points in the plane. Prove that there exists a subset of P , say the set P' of size $m = |P'| = \Omega(\sqrt{n})$ points, with the following property. The points of P' can be ordered, denoted by the sequence $\langle p_1, \dots, p_m \rangle$ such that the x -coordinate of the point p_i is greater than that of the point p_{i-1} , for all $i = 2 \dots m$. And additionally one of these is true: either the y -coordinate of each point p_i is greater or equal to that of p_{i-1} for all i . Or the y -coordinate of each point p_i is less than that of p_{i-1} for all i .
2. Let \mathcal{F} be a family of subsets of a n -element set. Prove that if \mathcal{F} is intersecting, then $|\mathcal{F}| \leq 2^{n-1}$. Is this the best bound? If so, can you give the corresponding example.
3. Let $n \leq 2k$ and A_1, \dots, A_m be subsets of size k of $A = \{1, \dots, n\}$, with the following property: $A_i \cup A_j \neq A$ for all i, j . Show that $m \leq (1 - \frac{k}{n}) \binom{n}{k}$. (Hint: Think of the complement of each set).
4. Given an integer k , let P be a set of n points such that each point has at least k points equi-distant from it. Assume no three points lie on the same line. Show that $k = O(\sqrt{n})$.
5. Prove that the graph obtained from K_n by deleting one edge has exactly $(n-2)n^{n-3}$ spanning trees.

Bonus Problem. Five couples are at a party, and each person shakes hands with some of the other people, but obviously does not shake hands with their own partner. Say one of the couples is Alice and Bob. Alice then asks each of the other 9 people how many times they shook hands, and receives all distinct answers. How many people did Alice's partner Bob shake hands with?

10 points.