Introduction to Combinatorics Spring, 2011

Homework 3 – Parity and double-counting arguments

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Questions

1. Prove that in any convex quadilateral, there exists a diagonal of length at least one-fourth the total perimeter.
2. The numbers 1 to 10 are arranged in an arbitrary order around a circle. Show that there are three consecutive numbers whose sum is at least 17.
3. In a contest with 3 problems, 80% of the students solve problem 1, 75% problem 2, and 70% problem 3. Prove that at least 25% of the students solve all three problems.
4. Given any set P of n points on the real line, and m distinct intervals on them (i.e., each interval's endpoints are from P), prove that there exists a point contained in m^2/n^2 intervals.
5. Given a set L of n lines in the plane (n is even, assume no three lines intersect at a point), show for that for any fixed point p, there exists a line l through p such that it intersects $n/2$ lines of L on both sides from p.
6. There are 30 senators in a senate. For each pair of senators, the two senators are either friends of each other or enemies of each other. Every senator has exactly six enemies. Every three senators form a committee. Find the total number of committees whose members are either all friends or all enemies of each other.

Bonus Problem. Given a continuous function $f : \mathbb{S}^1 \to \mathbb{R}$ ($\mathbb{S}^1$ is the one-dimensional sphere, i.e., a circle), and any two points $p, q \in \mathbb{S}^1$, prove that one can always rotate the two points p and q around $\mathbb{S}^1$ (without changing their position relative to each other) to get the points p' and q' such that $f(p') = f(q')$.

10 points.