Introduction to Combinatorics Spring, 2011

Homework 11 – Linear Algebra Method

Janos Pach & Nabil Mustafa *12 May*

Questions

1. Prove the "reverse oddtown theorem": given a set X of n elements, and m sets $S_1, \ldots, S_m$ over X where each $|S_i|$ is even, and $|S_i \cap S_j|$ is odd, prove that $m \leq n + 1$. Can you improve this to $m \leq n$?

2. Prove the "bipartite oddtown theorem": given a set X of n elements, and sets $R_1, \ldots, R_m$ and $B_1, \ldots, B_m$, where $|R_i \cap B_i|$ is odd for every i, and $|R_i \cap B_j|$ is even for every $i \neq j$, prove that $m \leq n$.

3. A block-design consisting of n total elements X and m sets over X where each set has k elements and every t-sized subset of X is contained in exactly λ sets is denoted by t-(n, k, λ). Then Fisher's inequality states that $m \geq n$.

   Prove that for a block-design with $t = 2$, $\lambda \cdot \frac{n-1}{k-1}$ has to be an integer.

4. Can there be a block-design of type 2-$(16, 6, 1)$? What about 2-$(21, 6, 1)$? 2-$(25, 10, 3)$?

5. For $\lambda = 1$, prove Fisher's inequality directly.

Bonus Problem. Suppose we have a necklace of n beads. Each bead is labelled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \ldots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k - 1 \quad \forall\, k = 1, 2, \ldots, n$$

10 points.