Introduction to Combinatorics Spring, 2011

Homework 10 – Probabilistic Methods

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Questions

1. Let X be a random variable, and $c > 0$ any constant. Prove that $Var[cX] = c^2 Var[X]$.
2. Prove that the Crossing Lemma is optimal. In other words, given any integers $n > 0$ and $m \geq 5n$, show that there exists a graph G with n vertices and m edges such that the crossing number of G is at most $c \cdot m^3/n^2$, where $c > 0$ is a constant.
3. In class we saw a probabilistic proof of the Crossing Lemma. Using that for intuition, construct a purely combinatorial double-counting proof of the Crossing Lemma.
4. In a way similar to the one done in class, prove that the number of incidences between n distinct unit circles and n distinct points in the plane is at most $c \cdot n^{4/3}$, where $c > 0$ is a constant.

Bonus Problem. A deck of 50 cards contains two cards labeled i for each $i = 1, 2, \ldots, 25$. There are 25 people seated at a table, each holding two of the cards in this deck. Each minute every person passes the lower-numbered card of the two they are holding to the right. Prove that eventually someone has two cards of the same number.

10 points.