Introduction to Combinatorics Spring, 2011

Homework 1 – Basic Counting

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Questions

1. Suppose that n teams play a tournament in which every team plays every other team exactly once, and there are no tie games. Prove by induction that, no matter what the individual game outcomes are, one can always number the teams $t_1, t_2, \ldots, t_n$, so that t_1 beat t_2, t_2 beat t_3, and so on, through t_{n-1} beat t_n.

2. How many ordered pairs (A, B), where A, B are subsets of $\{1, 2, \ldots, n\}$, are there such that $|A \cap B| = 1$?

3. Consider a set A of n elements. Count the number of sequences of the form $(A_1, A_2, \ldots, A_k)$ where $A_i \subseteq A_{i+1}$ and $A_k \subseteq A$.

4. Prove the following with a combinatorial argument:

$$\begin{aligned}
\binom{n+m}{2} &= \binom{n}{2} + \binom{m}{2} + nm \\
\sum_{k=0}^{n} \binom{x}{k} \cdot \binom{y}{n-k} &= \binom{x+y}{n} \\
\binom{\binom{n}{2}}{2} &= 3 \cdot \binom{n}{4} + 3 \cdot \binom{n}{3} \\
\sum_{k} \binom{n}{2k} \cdot \binom{2k}{k} \cdot 2^{n-2k} &= \binom{2n}{n}
\end{aligned}$$

5. Prove that the coefficient of x^k in $(1 + x + x^2 + x^3)^n$ is:

$$\sum_{i=0}^{k} \binom{n}{i} \cdot \binom{n}{k-2i}$$

6. Prove the following identity in three different ways, with a combinatorial argument, an inductive argument and using the binomial theorem:

$$\sum_{i=0}^{n} \binom{n}{i} \cdot 2^i = 3^n$$

Bonus Problem. Two players A and B have to divide a apple-pie among themselves. B divides the apple-pie into $2n$ (possibly unequal) slices, where n is any positive integer. Then A picks any slice from the $2n$ slices for himself. Then B picks one of the two slices which were next to A's slice. Then A picks a slice, then B and so on. The only restriction is that each time A or B pick a slice, the remaining apple-pie must consist of consecutive slices (i.e., all the slices picked so far must be consecutive). Prove that no matter how B divides the apple-pie, A can always get at least half the apple-pie for himself.

10 points.