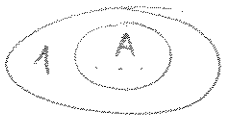



Combinatorial Counting

- # subsets of an n -element set is 2^n . At each element one has two choices, choosing it or not: $\overset{\vee}{1}, \overset{\vee}{2}, \overset{\vee}{3}, \dots$
- # subsets of an n -element set with an even number of elements is $2^{n-1} = \frac{1}{2} 2^n \Leftrightarrow$ Probability that a random subset of $\{1, \dots, n\}$ has an even number of elements is $\frac{1}{2}$.

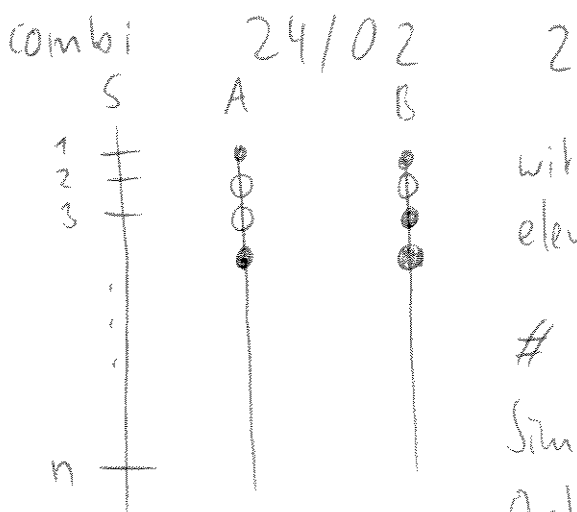
Proof:  $\forall A \subseteq \{2, \dots, n-1\}$
 A $A \cup \{1\}$
 odd \Rightarrow even

- What is the probability that $A \subseteq B$ or $B \subseteq A$, when A, B are randomly and independently chosen subsets of $\{1, \dots, n\}$?

Proof: We need to count # pairs (A, B) s.t. $A \subseteq B$ or $B \subseteq A$.
 Let A be such that $|A| = k$, then there are 2^{n-k} sets B that contain A , and there are 2^k B s.t. $B \subseteq A$.

 But there is one double counting so # pairs $(A, B) = 2^k + 2^{n-k} - 1$. Now all different possibilities need to be summed up:
$$N = \sum_{k=0}^n \binom{n}{k} (2^k + 2^{n-k} - 1) \square$$

So the sought for probability is $p = \frac{N}{2^n \cdot 2^n}$. An alternative way to derive N is the following.



with $A \subseteq B$. So there are for each element only three possibilities, hence

pairs (A, B) s.t. $A \subseteq B$ is 3^n .

Similarly # pairs (A, B) s.t. $B \subseteq A$ is 3^n .

Only equality must be considered now, hence

pairs (A, B) s.t. $A \subseteq B$ or $B \subseteq A$ is $2 \cdot 3^n - 2^n = N$.

Thus
$$p = \frac{2 \cdot 3^n - 2^n}{2^n \cdot 2^n} \xrightarrow{n \rightarrow \infty} \lim_{n \rightarrow \infty} 2 \left(\frac{3}{4} \right)^n.$$

So we proved by double counting that

$$\sum_{k=0}^n \binom{n}{k} (2^k + 2^{n-k} - 1) = N = 2 \cdot 3^n - 2^n$$

a purely combinatorial proof.

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Exam: Problems from problem sheets and proofs/theorems from lectures. Bonus problems can be handed in.

$\binom{n}{k} = \binom{n}{n-k}$ can be proved combinatorially as well, by identifying a 1-1 correspondence of subsets.

Binomial Theorem

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

as one needs to choose the 1 from all other brackets

proof: $(1+x)(1+x)\dots(1+x) = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots$

From every n bracket one can choose a factor. \nwarrow two brackets from where we choose x , rest 1.

combi 24/02 3

Corollary: Set $x = 1$, then $2^n = \sum_{k=0}^n \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$

Proof: $2^n = \# \text{ subsets} = \# 0 \text{ elem. subset} + \# 1 \text{ elem. subsets} + \dots \quad \square$

Corollary: $x = -1 : 0 = \sum_{k=0}^n \binom{n}{k} (-1)^k$

Adding the line for $x = 1$, and $x = -1$ gives:

$2^n = 2 \left(\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots \right)$, another proof of the statement on the first page, that $\# \text{ even element subsets} = \frac{2^n}{2}$

Proposition: $\sum_{k=0}^n \binom{n}{k} (2^k - 2^{n-k}) = 2 \cdot 3^n - 2^n$

Proof: By the binomial theorem, $\sum_{k=0}^n \binom{n}{k} 2^k = 3^n$. Also,

$$2^n \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{2}\right)^k = \left(1 + \frac{1}{2}\right)^n 2^n = 3^n. \text{ Finally}$$

$$\sum_{k=0}^n \binom{n}{k} = 2^n. \quad \square \quad (\text{Algebraic proof.})$$

What is the $\#$ different ways one can distribute k identical coins among n people (restricted case: everyone can get 0 or 1 coin) $k \leq n \Rightarrow \binom{n}{k}$.

Prop.: $\binom{n+k-1}{k}$

Proof: $\bullet \bullet \bullet \dots$ coins
1 2 3 ... n people
Sequence of length $n+k-1$, with separators: $\bullet \bullet | \bullet | \bullet \bullet | \bullet | \bullet \bullet \bullet$, $n-1$ separators, k coins. \square

ways one can write k as the sum of n non-negative integers = $\binom{n+k-1}{k}$. i.e. $k = x_1 + x_2 + \dots + x_n$, $x_i \geq 0$, integers.

Estimating $n!$

$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ for big n . Necessary:

$\lim_{n \rightarrow \infty} \frac{n!}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = 1$, (Stirling). Let's consider other estimates

$n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n < n^{n-1}$, assume $n = \text{even}$.

$$1 \cdot 2 \cdot 3 \cdot \dots \cdot \frac{n}{2} \cdot \left(\frac{n}{2} + 1 \right) \cdot \dots \cdot n$$

$$\frac{n}{2} \cdot \frac{n}{2} \cdot \frac{n}{2} \cdot \dots \cdot \frac{n}{2} \mid n \cdot \dots \cdot n = \left(\frac{n}{2}\right)^{n/2} \cdot n^{n/2} = \frac{n^n}{(\sqrt{2})^n}, \text{ a non-trivial upper estimate.}$$

$$n! = \underset{\substack{\uparrow \uparrow \uparrow \\ 1 \ 1 \ 1}}{1 \cdot 2 \cdot 3 \cdot \dots \cdot \frac{n}{2}} \cdot \underset{\substack{\uparrow \uparrow \\ \frac{n}{2} \ \frac{n}{2}}}{\left(\frac{n}{2} + 1\right) \cdot \dots \cdot n} \geq \left(\frac{n}{2}\right)^{n/2} > \frac{n^{n/2}}{(\sqrt{2})^n}$$

Estimates for $n!$: $\left(\frac{n}{2}\right)^{\frac{n}{2}} \leq n! \leq \left(\frac{n}{\sqrt{2}}\right)^n$

Stirling : $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$

means $\lim_{n \rightarrow \infty} \frac{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n}{n!} = 1$

Thm. : $e \left(\frac{n}{e}\right)^n \leq n! \leq en \left(\frac{n}{e}\right)^n$

Proof: let us first estimate the natural logarithm of $n!$

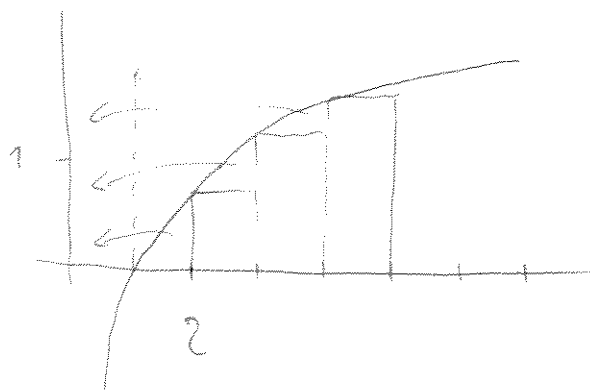
$\log n! = \log 1 + \log 2 + \dots + \log n$

$\log(n!) \leq \int_1^{n+1} \log x dx = x \log x - x$

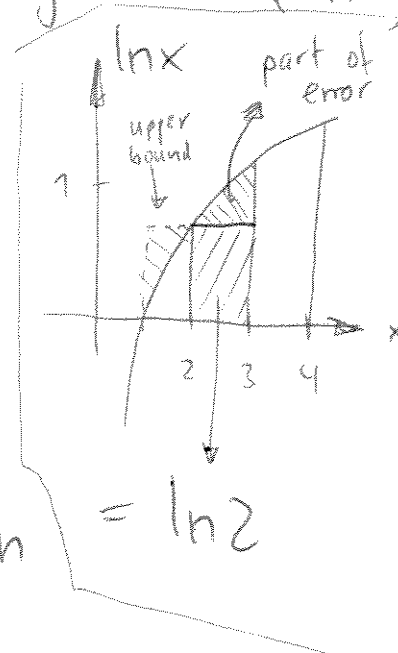
$= (n+1) \log(n+1) - (n+1) + 1 = (n+1) \log(n+1) - n$

How big is the error?

Look at the strip from zero to one and shift the error "triangles" into the strip:



An alternative is to consider an upper bound of the error by looking at rectangles as sketched above.



$$\text{Then } n! \leq e^{(n+1)\ln(n+1) - n} = \frac{(n+1)^{n+1}}{e^n} \quad \forall n \in \mathbb{N}$$

$$(n-1)! \leq \frac{n^n}{e^{n-1}}$$

$$\Rightarrow n! = n \cdot (n-1)! \leq n e \left(\frac{n}{e}\right)^n \quad \square$$

Note: "Integral method": $f(1) + f(2) + \dots + f(n) \leq \int_1^{n+1} f(x) dx$
whenever $f(x)$ is monotonely increasing.

$$\text{Thm.: } \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

Mnemonic:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

$$\text{Proof: } \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

$$\approx \frac{n^k}{(k/e)^k} = \left(\frac{en}{k}\right)^k$$

Include an e from first thm. to make this proof correct.

Recall the binomial theorem,

$$\binom{n}{0} + \binom{n}{1}x + \dots + \binom{n}{k}x^k \leq (1+x)^n, \quad 0 < x < 1$$

Dividing both sides by x^k gives:

$$\underbrace{x^{-k} + n x^{1-k} + \dots + \binom{n}{k}} \leq \frac{(1+x)^n}{x^k}$$

each coefficient x^{1-k}
is larger than one,
thus

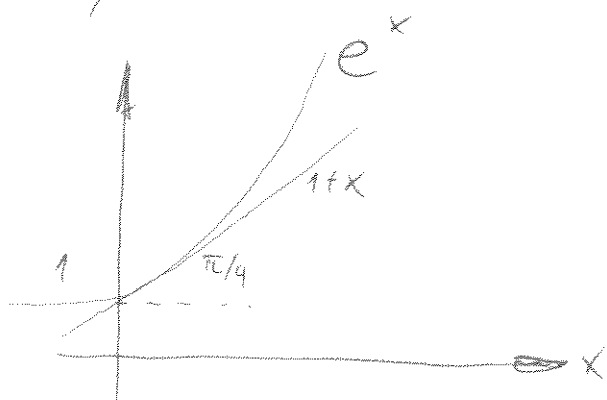
$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{k} \leq \frac{(1+x)^n}{x^k} \quad \forall 0 < x < 1$$

Let $x = \frac{k}{n} < 1$ as $k < n$, then

$$\sum_{i=0}^k \binom{n}{i} \leq \frac{(1 + \frac{k}{n})^n}{(\frac{k}{n})^k} \leq \frac{(e^{\frac{k}{n}})^n}{(\frac{k}{n})^k} \quad \text{since}$$

$$1+x \leq e^x \quad \forall x > 0$$

i.e. e^x is convex, i.e. always above its tangent line. \square



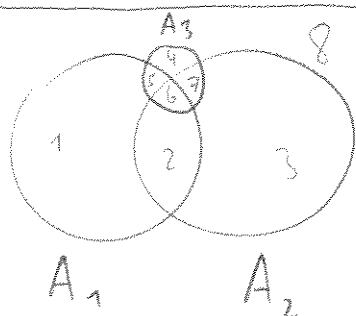
Here are another methods for estimating sequences

Generating Function Method

$$a_0 + a_1 + \dots + a_k \xleftrightarrow{\text{map}} a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k := f(x)$$

Taking derivatives gives coefficient.

Inclusion/Exclusion Formula (Principle of I/E \rightarrow PIE)



$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

So there are 8 regions an element can be.

Only A_i , $A_i \cap A_j$, $A_1 \cap A_2 \cap A_3$, or in none. So we can check the above formula by comparing the regions covered on each side.

combi 03/03 4

Notation: $\binom{|X|}{2} = \#$ 2 element subsets of set X
 $\binom{X}{2} = \{2\text{-element subsets}\}$

For any n :

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{i=1}^n |A_i| - \sum_{\{i,j\} \in \binom{\{1,2,\dots,n\}}{2}} |A_i \cap A_j|$$

$$+ \sum_{\{i,j,k\} \in \binom{\{1,2,\dots,n\}}{3}} |A_i \cap A_j \cap A_k|$$

$$- \sum_{\{i,j,k,l\} \in \binom{\{1,2,\dots,n\}}{4}} |A_i \cap A_j \cap A_k \cap A_l| + \dots - \dots + \boxed{\text{shaded}} (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n|$$

Proof: Every element that belongs to the union of the sets of the LHS can be counted only once on the RHS.

Pick an element that belongs to precisely t of the sets A_i , $i \in \{1, \dots, n\}$, $t \in \{t \geq 1\}$.

↓ counted once on LHS

$$1 = t - \binom{t}{2} + \binom{t}{3} - \binom{t}{4} + \dots + (-1)^{t+1} \binom{t}{t}$$

which holds by the binomial formula:

$$1 - t + \binom{t}{2} - \binom{t}{3} + \dots = (1-1)^t = 0. \quad \square$$

What is the probability that if we put n letters randomly into n envelopes then at least one letter is put in the right envelope? $\approx \frac{1}{e}$

Proof: Envelopes 1 2 3 ... n $n!$ possibilities,
 letters 5 10 3 ... 2

Let $A_3 = \{ \text{permutations where the third letter gets into the third envelope} \}$.

Then $|A_1 \cup A_2 \cup \dots \cup A_n|$ is # permutations where at least one letter ends up at the right place.

$$\left| \bigcup_{i=1}^n A_i \right| \stackrel{\text{PIE}}{=} n(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \dots$$

$$= n(n-1)! - \frac{n!}{2!} + \frac{n!}{3!} - \frac{n!}{4!} + \dots = n! \sum_{k=1}^n \frac{(-1)^{k+1}}{k!}$$

$$\sum_{k=1}^n \frac{(-1)^{k+1}}{k!}$$

$n \rightarrow \infty$

\rightarrow

$$\frac{1}{e}$$

Taylor: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ \square

\square

Lemma: Handshake - Lemma

The number of people who shook an odd number of hands is even.

Proof:



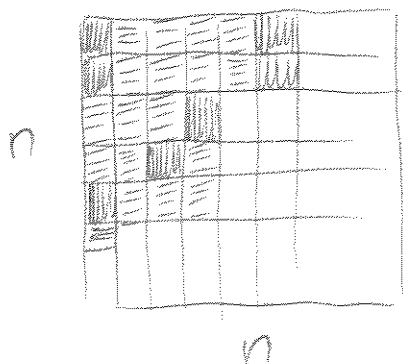
handshakes = $\deg(v)$.

$$\sum_{v \in V(G)} \deg(v) = 2 |E(G)| \quad \square$$

This was an example for a parity argument.

Extra bonus problem:

Let there be people living on a $n \times n$ grid, some of them infected. Each person has at most eight neighbors.



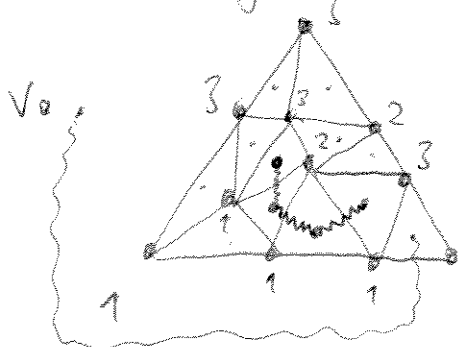
Rule 1: If ≥ 2 neighbors of a square are infected then the square gets infected

Rule 2: Infection is eternal.

- ① What is smallest # people that one can infect all squares?
- ② Same game but neighbor is defined differently. Every square has at most 4 neighbors.

Lemma: (Sperner)

Having vertices on edges of big triangle and inside it, connect them maximally such that no vertex pair crosses each other.



The vertices are coloured in three different colours. (2,3 respectively)

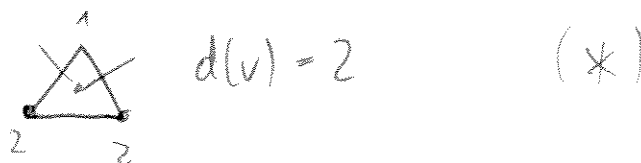
2 On the side opposite to 1st only vertices with colour 2 and 3 are allowed.

But all the other vertices can be coloured arbitrarily.

Sperner's lemma says, then \exists a small triangle which has all three colours.

Proof: Put a vertex into each face of the small triangles and one outside the big triangle and connect two of those new vertices iff they are neighbors (they share a side) and ~~they share a~~ this side is of type ①—②.

This defines a new graph, say G . For the sake of contradiction, suppose \nexists a small three coloured triangle. Then $d(v)$ is even for every vertex v , except for v_0 ,



and $d(v_0) = \text{odd}$. This can be seen when walking along an outer edge then there must be an odd number of colour changes. When there is a colour change $\textcircled{1} \leftrightarrow \textcircled{2}$ then there is an edge connecting to the outside v_0 .

Now $\sum d(v) = \text{even} \Rightarrow \exists$ internal vertex with odd degree, contradicting $(*)$. ~~*~~

Remark: One could generalise this lemma to higher dimensions. Consider for example a tetrahedron, where one can also find a small internal tetrahedron, having four ~~out~~ coloured vertices.

Thm.: (Brouwer's fixed point theorem)

Let $f: [0, 1] \rightarrow [0, 1]$ be a continuous function.

Then $\exists x_0 \in [0, 1]$ s.t. $f(x_0) = x_0$.

Proof: If $f(0) \neq 0$ then $f(0) > 0$, if $f(1) \neq 1 \Rightarrow f(1) < 1$ and let $g(x) := f(x) - x$ then we get for the two cases: $g(0) = f(0) - 0 > 0$ and $g(1) = f(1) - 1 < 0$
 $\Rightarrow \exists x_0$ s.t. $g(x_0) = 0$ by Rolle. \square

combi 10/03 3

In higher dimensions, the theorem is more difficult:

Thm: Let f be a continuous mapping from one triangle to another, then $f(p) = p$ for some p .



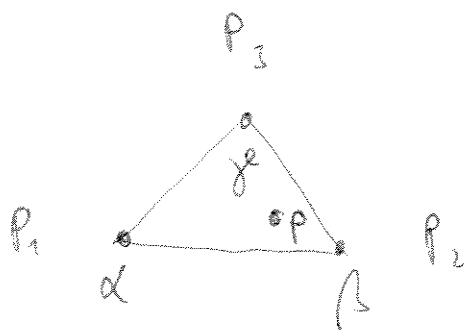
(The shape does actually not matter so the regions only need to be simply connected).

Def.: Barycentric coordinates in the plane.

Assign to each point three coordinates,

$$p = (\alpha(p), \beta(p), \gamma(p)) \text{ such that } \alpha + \beta + \gamma = 1.$$

Fix three points of the plane and allocate a fraction of the unit mass to the three points. So we define:



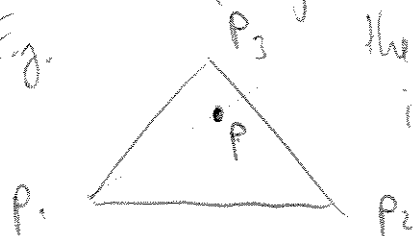
$$x(p) = \alpha \cdot x(P_1) + \beta \cdot x(P_2) + \gamma \cdot x(P_3)$$

$$y(p) = \alpha \cdot y(P_1) + \beta \cdot y(P_2) + \gamma \cdot y(P_3)$$

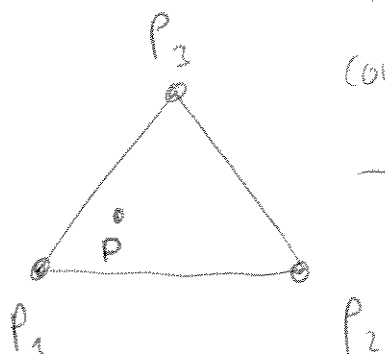
p is the centre of gravity of the triangle.

So every point p inside the triangle defined by P_1, P_2, P_3 has 3 coordinates $\alpha(p), \beta(p), \gamma(p)$ such that the center of gravity of P_1, P_2, P_3 with these weights is p .

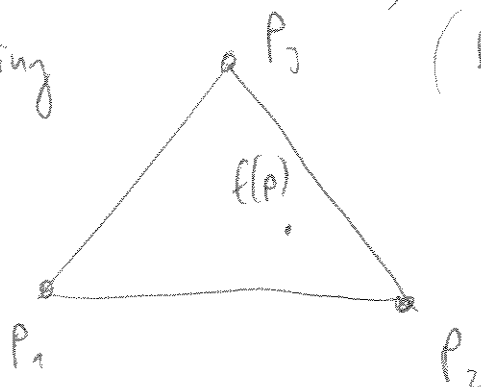
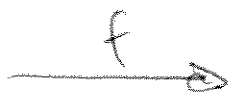
E.g. then I can choose α, β, γ such that p is in the center of gravity of P_1, P_2, P_3 .



Proof: (Brouwer's fix-point theorem in 2D)



continuous mapping



($f(p)$ can be on the boundary.)

Define a colouring of all $p \in \Delta$ in the following way:

p has color 1 if $\alpha(f(p)) < \alpha(p)$

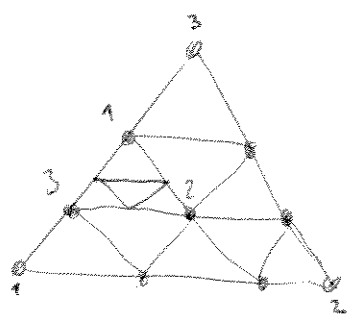
(intuitively, $f(p)$ is further away from p_1 than p)

p has color 2 if $\beta(f(p)) < \beta(p)$

p has color 3 if $\gamma(f(p)) < \gamma(p)$.

(Each point gets at least one color)

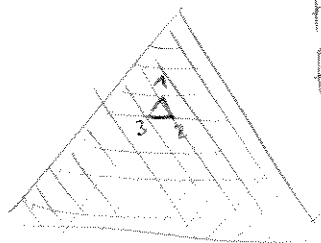
If f has no fixed point, i.e. $f(p) \neq p$



Clearly, $P_1 = (\alpha=1, \beta=0, \gamma=0)$,

and Sperner's lemma applies, since on an edge there cannot be a vertex with colour of the opposing big triangle vertex.

Consider the same triangle with a finer mesh, apply Sperner's lemma again and maybe the tri-color triangle is somewhere else. When making the mesh finer and finer then there is a point of accumulation of the small multicolored triangles. Call this point p . This is the limit of a sequence of points colored in 1, 2, 3, respectively.



combi 10/03 5

Since a sequence $p_1^i, p_2^i, p_3^i, \dots$ s.t. $\alpha(p_1^i) > \alpha(f(p_1^i))$

$$\Rightarrow \alpha(f(p)) \leq \alpha(p).$$

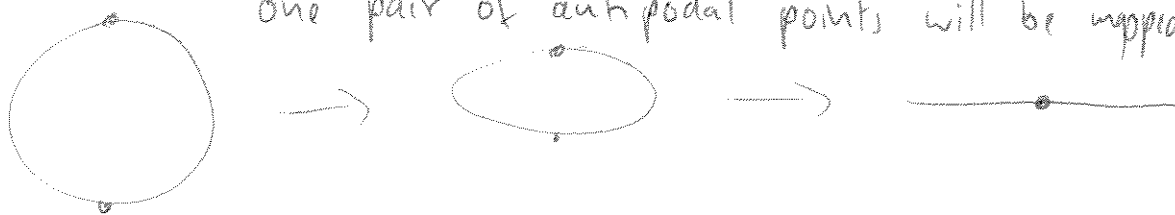
And also $\beta(f(p_2^i)) < \beta(p_2^i)$ and $\gamma(f(p_3^i)) < \gamma(p_3^i)$
for corresponding sequences.

But $\alpha(f(p)) + \beta(f(p)) + \gamma(f(p)) = 1$ and also

$$\alpha(p) + \beta(p) + \gamma(p) = 1 \text{ which implies}$$

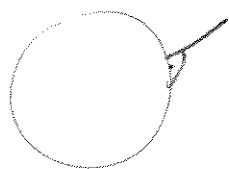
that $f(p) = p$. \square

Remark: There are equivalent versions of Brouwer's fix-point theorem. So when squeezing a spherical shell into the plane, one pair of antipodal points will be mapped together.



$$f: S^2 \rightarrow \mathbb{R}^2; \quad f(p) = f(p^*)$$

Theorem: (Hedgehog theorem) One cannot comb a hedgehog.

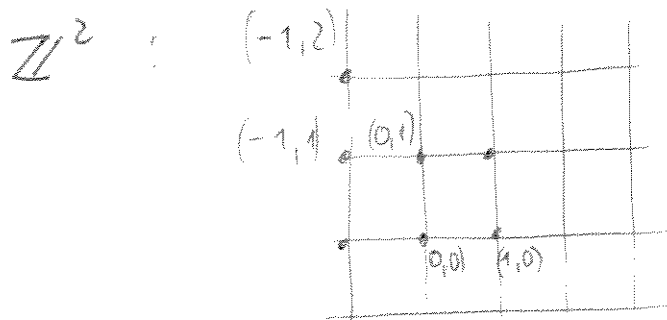


Assumptions: Continuity and no hair sticks perpendicularly out from the surface.

Remark: Two dimensional hedgehogs can be combed:



Recap exercise: (odd, even)



Show that no matter how we select 5 points, there will be 2 among them such that the midpoint of the line joining them is also an integer point.

Proof: This can be solved via parity and pigeon hole principle. Pigeon hole implies, given the five points, at least two must have the same parity, for example $(a, b), (a', b')$ s.t. $a \equiv a' \pmod 2$ and $b \equiv b' \pmod 2$ hence their mid-point $(\frac{a+a'}{2}, \frac{b+b'}{2})$ must be $\in \mathbb{Z}^2$.

Let us consider a "fractional" version of the pigeon hole principle. Given n glasses and 1l of liquid, there is at least one glass with $\geq \frac{1}{n}$ liter

$\lfloor \quad \rfloor \quad \lfloor \quad \rfloor \quad \dots \quad \lfloor \quad \rfloor$ and one with $\leq \frac{1}{n}$ liter.

n glasses

E.g.: Consider an $m \times k$ matrix (a_{ij}) :

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & \dots & & a_{mk} \end{matrix} \leftarrow \text{average value of row 1: } \frac{\sum_{i=1}^k a_{1i}}{k}$$

Assume the average of each row is at least one hundred, then there must be a column with average at least 100.

combi 17/03 2

$$a_{11} + a_{12} + \dots + a_{1k} \geq 100k$$

$$a_{21} + a_{22} + \dots + a_{2k} \geq 100k$$

\vdots

$$a_{m1} + a_{m2} + \dots + a_{mk} \geq 100k$$

$$\sum_{i=1}^m a_{i1} + \sum_{i=1}^m \dots \geq 100mk$$

By contradiction, \exists one column with sum $\geq 100m$.

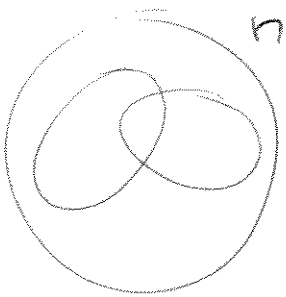
This was another example of double counting, since

$$\sum_{i=1}^m \sum_{j=1}^k a_{ij} = \sum_{j=1}^k \sum_{i=1}^m a_{ij}$$

~~Thm.: (Extremal set theorem)~~

Extremal Set Theory

Thm.: (Erdős-Ko-Rado theorem)



Let there be n elements, $S = \{1, \dots, n\}$, and k -element subsets. What is the maximal # distinct k -subsets?

E.g.: For $k=2$, how many edges can we maximally pick among n points such that any ed two points has one in common.
point.

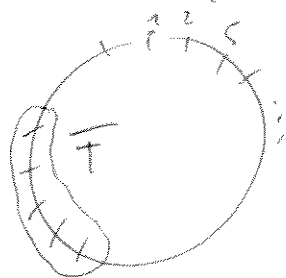
Thm.: (Erdős-Ko-Rado) intersecting

If $n \geq 2k$ and \mathcal{F} is a system of k -element subsets of X where $|X| = n$, then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

\mathcal{F} is called an intersecting system, i.e. $F \cap F' \neq \emptyset$
 $\forall F, F' \in \mathcal{F}$.

Proof: By double counting. Assume \mathcal{F} is intersecting, containing k elements ~~each~~ subsets of $X = \{1, \dots, n\}$, i.e. $\forall F \in \mathcal{F}, |F| = k$.



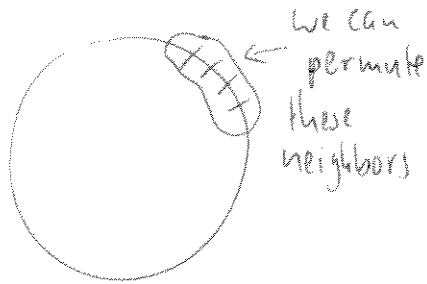
How many ways are there to place n numbers on a circle?

$$\# \text{ seatings} = (n-1)!$$

$F \in \mathcal{F}$ shall be said to be consecutive in a seating s , if the elements of F are sitting next to each other, i.e. form an interval.

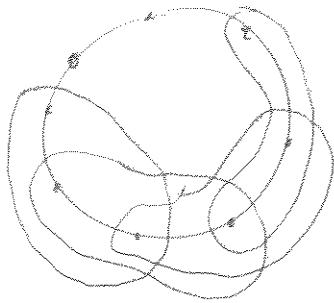
We double count the number of pairs (s, F) where s is a seating and F a consecutive element of \mathcal{F} with respect to the seating s .

$$\# (s, F) = \sum_{F \in \mathcal{F}} (\# \text{ seatings in which } F \text{ is consecutive})$$



$$\sum_{F \in \mathcal{F}} \underbrace{(\# \text{ seatings in which } F \text{ is consecutive})}_{k! (n-k)!}$$

Given a seating:



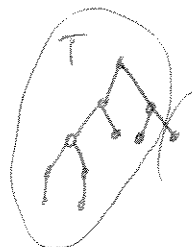
$$= \sum_{\substack{\text{seatings} \\ S}} \underbrace{(\# \text{ intervals of length } k \text{ that belong to } \mathcal{F})}_{\leq k, \text{ as they must be intersecting but disjoint}}$$

\uparrow
 $= (n-1)!$
 $\leq (n-1)! k$

Hence $|\mathcal{F}| k! (n-k)! \leq (n-1)! k$

$$|\mathcal{F}| \leq \frac{(n-1)! k}{(n-k)! k!} = \binom{n-1}{k-1} \quad \square$$

Def.: A tree is a simple, connected acyclic ^{finite} graph.



Prop.: Any tree on n vertices has n-1 edges

Proof: By induction on the number of vertices.

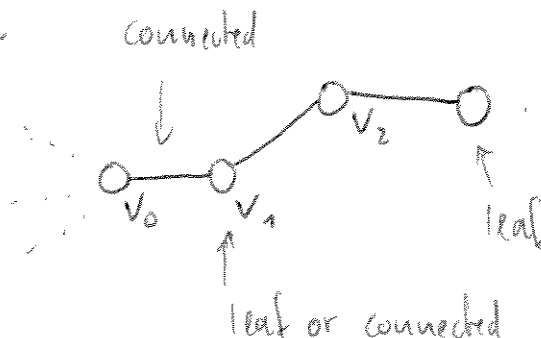
n=1



n=2



Claim: If $n \geq 2$ then every tree with n vertices has a vertex of degree 1 (a leaf). The proof is algorithmic.



Whenever you go to a new vertex it cannot be an already visited one as a tree is acyclic.

Suppose T has n vertices and the proposition is true for all trees with at most $n-1$ vertices. By claim, find a leaf via the above algorithm and delete it. So it remains to show that $T - \{\text{leaf}\}$ is a tree. This holds clearly. \square

Maxwell studied graphs for electrical networks, and inspired the following theorem:

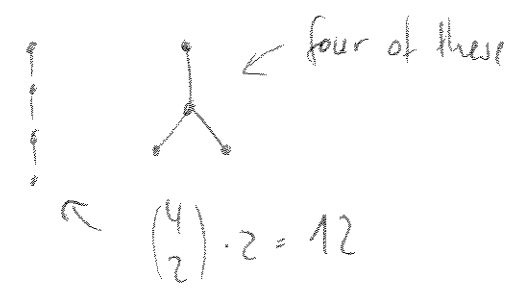
Thm.: (Cayley's thm.)

What is the # different trees that one can define on $n \geq 2$ labeled vertices? $f(n) = n^{n-2}$.

E.g. $\begin{matrix} 1 & 2 & 3 \\ \text{---} & \text{---} & \end{matrix} \neq \begin{matrix} 1 & 3 & 2 \\ \text{---} & \text{---} & \end{matrix}$. (Two graphs are the same if for each vertex the list of the neighbors is the same).

Proof: For $n=3$, $f(3) = 3$.

$n=4$ it is already harder:



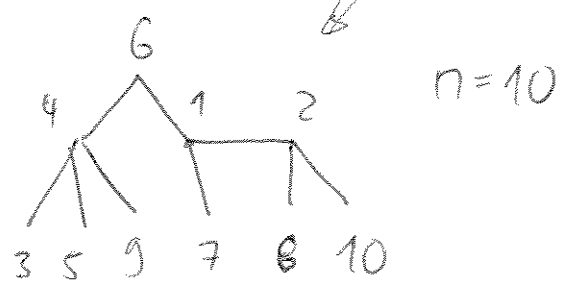
$f(4) = 16$.

It is to show that $f(n) = n^{n-2}$.

has code 44124612

(Prüfer's proof)

Assign a code word to each tree which has length $n-2$ and



its letters will be numbers between 1 and n and

Combi 17/03 6

We can repeat the same number. Clearly $\# \text{ trees} \leq \# \text{ code word}$
 How to see the one-to-one correspondence between the codes and the trees? $\underbrace{\quad}_{n^{n-2}}$

Consider the following algorithm to form a code for a given tree.
 Pick the leaf v with the smallest label. This leaf has a unique neighbor and write down its label. Next delete the leaf v and recurse to get a code word. Applied to our example: 44124612
 Stop when only two vertices left. $10-2=8$

To show 1-1 correspondence we need to show that the code is enough to reconstruct the tree in a unique way. I.e. no two trees have the same code.

Proposition: Let $d_i := \text{degree of vertex } i \text{ in } T$, then
 $d_i = (\# \text{ occurrences of } i \text{ in the code}) + 1$.

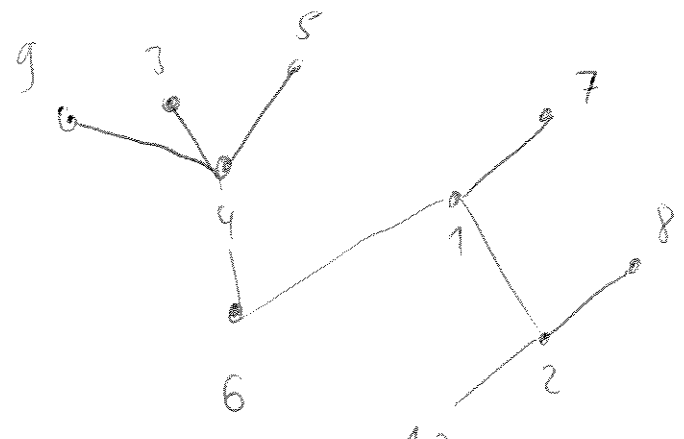
For example $d_1=3, d_2=3, d_3=1, d_4=4, d_5=1, d_6=2$
 $d_7=1, d_8=1, d_9=1, d_{10}=1$, for code 44124612

As a check, $\sum_{v \in T} d(v) = 2E = 2(n-1) = 18$ for $n=10$
 and a T_n tree.

Proof of proposition: ?

i	d_i
1	3 (2, 1)
2	3 (2, 1)
3	1
4	4 (3, 2, 1)
5	1
6	2
7	1
8	1
9	1
10	1

smallest leaf



One can generalise this. □

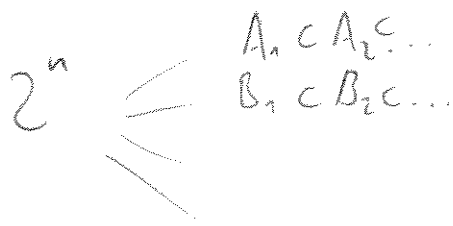
Combi 24/03 1 $\binom{n}{\frac{n}{2}} = \binom{n}{n-\frac{n}{2}}$ $\frac{n!}{(\frac{n}{2})!(\frac{n}{2})!} = \text{For odd } n?$

Thm.: (Sperner's theorem)

One can pick at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ subsets from $\{1, \dots, n\}$ such that no subset is contained in another one.

E.g.: $n=3$, $\{1, 2, 3\}$, there are 8 subsets. What's the maximal # subsets such that no set is completely contained in another. Here it is 3. $\{1\}, \{2\}, \{3\}$, for example.

Proof 1: Intuitive idea: Split all subsets into chains (partition) into t chains. As they are disjoint t must be equal to the maximal number of non-pairwise contained sets.



$A_1 \subset A_2 \subset A_3 \dots$ is a chain.

To find this partition is difficult. So we better count more chains, non-disjoint, and divide them then by to address doubly counted sets.

$\{1, 2, \dots, n\}$. Pick some permutation order and construct from it a chain in the following way: E.g. $\{2, 5, 3, 1, 4\} \rightarrow \{2\}, \{2, 5\}, \{2, 5, 3\}, \{2, 5, 3, 1\}, \{2, 5, 3, 1, 4\}$.

So we have $n!$ permutations and each chain has n elements. A fixed set S of k elements is contained in $k!(n-k)!$ chains.

Let $\mathcal{M} = \{M_1, \dots, M_k\}$, a set of sets where no set is a subset of another. We double count the following pairs: let $M' \in \mathcal{M}$ and S be a permutation s.t. M' was contained in the chain generated from permutation S . (For example with $n=3$ it is clear).

$$(\# \text{ such pairs}) \leq n!$$

Note that each permutation appears only once in each (M, S) . I.e. $(M, S), (M', S)$ is impossible, as $M \not\subseteq M' \wedge M' \not\subseteq M$. Let $|M'| = k$, then M' appears in exactly $k!(n-k)!$ pairs.

Thus

$$\sum_{i=1}^k |M_i|! (n - |M_i|)! = \# \text{ pairs } (M', S) \leq n!$$

$$\sum_{i=1}^k \frac{1}{\frac{|M_i|! (n - |M_i|)!}{n!}} \leq 1. \text{ Look at the binomial terms to see:}$$

$$\sum_{i=1}^k \frac{1}{\binom{n}{|M_i|}} \leq \sum_{i=1}^k \frac{1}{\binom{n}{|M_i|}}$$

$$\frac{k}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq 1 \Rightarrow k \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}. \quad \square$$

Proof 2: By induction.

To show: given any 2^n subsets of $\{1, \dots, n\}$,
one can partition them into $\binom{n}{\frac{n}{2}}$ chains.

These chains shall be symmetric, e.g.

$\{1, 2, 3\}$ has subsets of size 0, 1, 2, 3, 4,

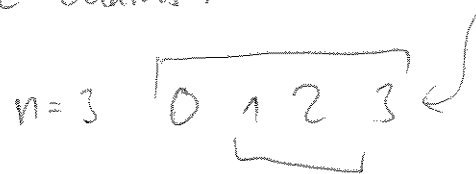
the symmetric

$\{1, 2, 3\}$ into $\binom{3}{\lfloor \frac{3}{2} \rfloor}$ symmetric chains:

$\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}$

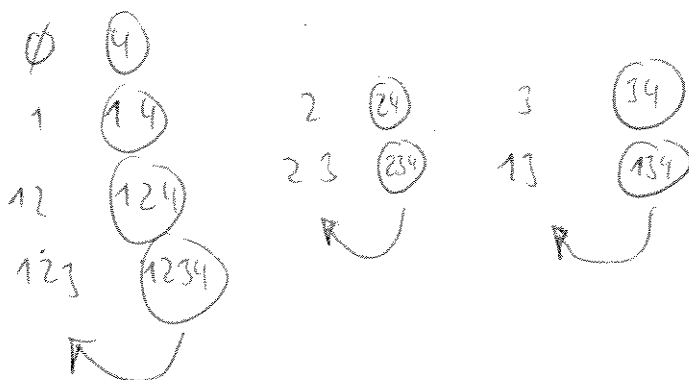
$\{2\}, \{2, 3\}$

$\{3\}, \{1, 3\}$.



Claim: one can always find such $\binom{n}{\frac{n}{2}}$ symmetric chains that partition a set with n elements.

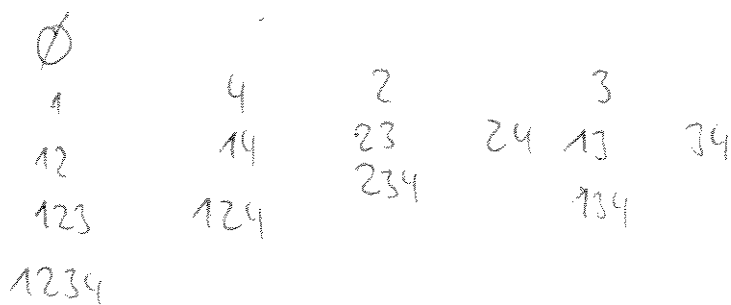
n=3. Now add 4 to each chain element.



There are $ab = \binom{4}{\frac{4}{2}}$

but they are not symmetric.

But one can easily make them all symmetric (arrows)



Now are all symmetric.

Use induction on n to complete the proof.

1...n

↓ Inductive hypothesis

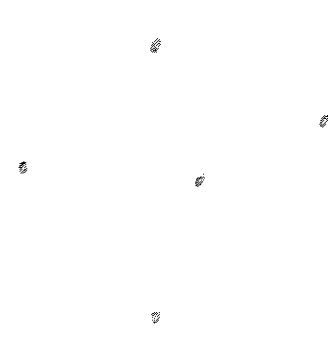
1...n-1 1... k... (n-1-k) ... n-1 n

1... k+1 ... (n-k) ... n-1 n

lots of initial chains will disappear in the symmetrizing process. \square

Unit Distances Problem

let there be n points in \mathbb{R}^2 . How many pairs ~~can~~ be ~~maximal~~ at unit distance can be created maximally by placing n points?



Is $2n$ pairs (by placing them on a grid) the maximal number? It is known that $n^{4/3}$ is an upper limit.

We'll show that there are at most $n^{3/2}$ pairs.

Case 1: There is a point with $\geq \sqrt{n}$ other points at unit distance. Pick \sqrt{n} of them, call this set P' .
 Claim: P' can only have $2n$ unit distances in total.
 Remove P' from the circle.



combi 31/03 1

Thm.: (Dilworth)

Given a partial order (X, \leq) ,

- i) If largest chain in X has size k , then X can be partitioned into k chains (\Rightarrow Erdős-Szeker)
- ii) If largest anti-chain in X has size k , then X can be partitioned into k chains. (\Rightarrow decomposition in $\binom{2}{\lfloor \frac{n}{2} \rfloor}$ chains)

E.g.: A sequence of n integers contains an increasing subsequence; for example

3 5 9 7 4 1 6 8 10 2
← ← ← ← ←
→ → → → →

Proposition: In any sequence of n elements one can find an increasing or decreasing subsequence of length $\frac{n}{2}$.

Disproof: That's false, since $n=10$, 8910 567 234 1
(1, 2, ..., \sqrt{n}) ($\sqrt{n}+1, \dots, 2\sqrt{n}$) (...)

Proposition: In any sequence of n elements one can find an increasing or decreasing subsequence of length \sqrt{n} .

Proof: $a_1, a_2, \dots, a_n, a_i \in \mathbb{N}$
Pick a_i and consider the largest increasing subsequence ending (and including) with a_i .

Combi 31/03 2

length of longest incre. ss. ending with b

For example:

	1	2	3	3	2	1	3	4	5	2
	3	5	9	7	4	1	6	8	10	2

Note that elements with the same upper number form a decreasing subsequence.

upper number

Case 1: There is an upper number $\geq \sqrt{n}$ ✓

Case 2: One upper number must be repeated at least $n/\sqrt{n} = \sqrt{n}$ times by the pigeon-hole principle. □

Recall: When one can compare every pair of a set, the set is totally ordered. A partial order \preceq must satisfy:

Transitivity, reflexivity and anti-symmetry

A set of elements with partial order is called a chain if every pair of elements can be compared. The set is an antichain, no pair can be compared.

Proof: (Dilworth)

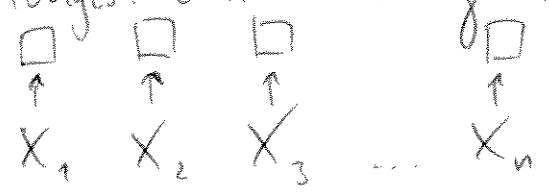
E.g. for a): Impose $a_i \preceq a_j$ iff $i \leq j$ and $a_i \leq a_j$ on the sequence problem. Then a chain is an increasing subsequence and an anti-chain a decreasing subsequence. The proposition follows.

E.g. for a): $X = \{1, \dots, n\}$, partial order $(2^X, \subseteq)$,
(last time)

- largest anti-chain has size $\binom{n}{\lfloor n/2 \rfloor}$.
- 2^X has a decomposition into $\binom{n}{\lfloor n/2 \rfloor}$ symmetric chains. ←

a) Let $X = \{x_1, \dots, x_n\}$ with partial order \preceq

For each x_i consider the length of the longest chain ending at x_i :



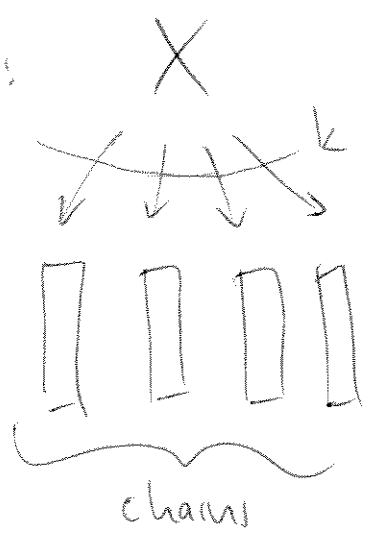
Claim: All elements with the same upper-number form an anti-chain.

(Since if two had the same label one could extend one to form a longer chain).

b) By induction.

Assume the largest anti-chain in X has length k

To show:



induction on n



Otherwise: There must be an anti-chain of length k .

Note, two elements of an antichain can not be in the same chain.

Assume the longest chain in X, C

(ϵ) There remain $X \setminus C$ elements.



Easy case: If largest anti-chain in $X \setminus C$ has size $k-1$. Induction then completes the proof (use a)!)
 ↑
 Induction on k



Now each chain element of C is comparable to one of \tilde{C} .

So one can divide the elements of C into two categories.
If



Transitivity forbids that t , a member of C , can be greater than one x_i and less than another x_j of \tilde{C} .

Problem 1: A or B could be empty. Impossible since else C would not have been the longest chain.

Problem 2: An element $t \in C$ cannot be in A and B because of transitivity.

Probabilistic Method

Book: Alon, Spencer, title ↑

At the beginning the probabilistic method seems to be merely a language to describe counting.

A discrete, such as $\Omega = \{1, \dots, 6\}$ for tossing a die, is finite.

Also $\sum_{\text{all } i} p(i) = 1$ and $0 \leq p(i) \leq 1 \quad \forall i$. An event would be for example $A = \text{"die comes up even"} = \{2, 4, 6\}$. $P(A) = \frac{1}{2} = P(B)$ where $B = \{1, 3, 5\}$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

An example for a random function would here be $f(i) = i^2$, i being the number of eyes of the die.

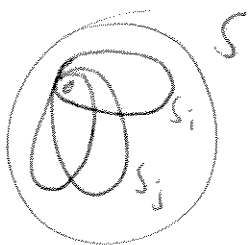
$$E[f] = \sum_{\text{all } i} p(i) f(i), \text{ a linear operation:}$$

$$E(\alpha g(x) + \beta f(x)) = \alpha E(g(x)) + \beta E(f(x))$$

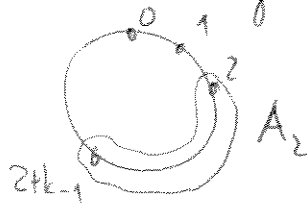
Thm.: (Erdős - Ko - Rado)

If \mathcal{F} is a family of k -element subsets of $S = \{1, \dots, n\}$ s.t. $\forall S_i, S_j \in \mathcal{F} \quad S_i \cap S_j = \emptyset$, then $n \geq 2k$

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$



Recall: In the double-counting proof we considered



Proof: $\forall i$ s.t. $0 \leq i \leq n-1$, $A_i = \{i, i+1, \dots, i+k-1\}$, $|A_i| = k$
mod n

Pick randomly and uniformly a permutation

$$\sigma : \{0, 1, \dots, n-1\} \rightarrow \{0, 1, \dots, n-1\}$$

and pick randomly and uniformly an integer $i \in \{0, 1, \dots, n-1\}$

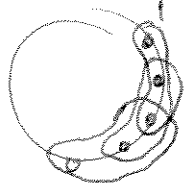
Consider A (underlined as it is a random set, depending on the two random variables i and σ),

$$\underline{A} = \{\sigma(i), \sigma(i+1), \dots, \sigma(i+k-1)\} \subseteq S \text{ where}$$

$$|\underline{A}| = k. \quad \Pr[\underline{A} \in \mathcal{F}] \leq \frac{k}{n} ?$$

Claim: # different A_i 's that belong to \mathcal{F} is $\leq k$,

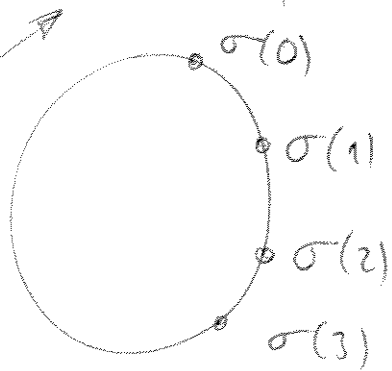
"conditioned on σ " since



for $k=3$.

$$(*) \Pr[\underline{A} \in \mathcal{F} \mid \sigma] \leq \frac{k}{n} \text{ for any } \sigma.$$

prbs. that starting point is i



since having chosen a starting point limits the number of consecutive sets.

Recall: Conditional probability



$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

combi 07/04 3

Because the probability that $A =$ any given k -element subset of S is the same,

$$\Pr[A \in \mathcal{F}] = \frac{|\mathcal{F}|}{\binom{n}{k}}$$

as for each given k -subset (of which there are $\binom{n}{k}$) the probability is $\frac{1}{\binom{n}{k}}$ to be chosen.

Comparing this with (*) gives $|\mathcal{F}| \leq \frac{k}{n} \cdot \binom{n}{k}$. \square

Proposition: Let B_1, \dots, B_k be disjoint events, $\sum_i \Pr(B_i) = 1$, then if $\Pr(A|B_i) \leq c \Rightarrow \Pr(A) \leq c$.

$$\begin{aligned} \text{Proof: } \Pr[A] &= \sum_{i=1}^k \Pr(A \cap B_i) = \sum_{i=1}^k \Pr(A|B_i) \Pr(B_i) \\ &\leq c \sum_{i=1}^k \Pr(B_i) = c. \quad \square \end{aligned}$$

All of the double-counting proofs can be translated into probabilistic language.

Def.: Tournament: Any two of n players play a match where there is no draw. If 1 beats 2, a directed edge is drawn:



i.e. a directed complete graph.

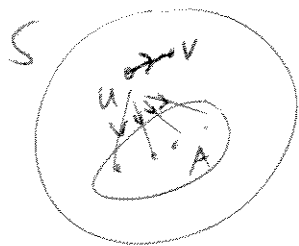
Schütte's problem: Does there exist a tournament (finite n) such that $\forall A \subseteq V(T), |A|=k$ there is a $v_A \in V(T)$ s.t. $\vec{v}_A u \in E(T) \forall u \in A$.
 i.e. for any k players there is one who beats all of them. (Property S_k).

Remark: $\underbrace{1 \ 2 \ 3 \ \dots}_k$ an infinite amount of players,
 one can always find a player who beats all of the others,

Proposition: (Erdős 1963) Yes, Schütte's proposition holds.

Thm.: $\forall n > k^2 2^k (\ln 2 + O(1)) \exists$ a tournament with n vertices that has property S_k .

Proof: $\forall A \subseteq S$ with $|A|=k, |S|=n$, define a random variable X_A . For each pair of vertices, let the outcome of their match have probability $\frac{1}{2}$. So there



are $2^{\binom{n}{2}}$ ways to draw the tournament, each is equally likely. So our probability space

consists of $2^{\binom{n}{2}}$ directed complete graphs on n vertices,

$$X_A := \begin{cases} 1 & \text{if for } A \exists v_A \text{ that beats all elements of } A \\ 0 & \text{else} \end{cases}$$

X_A is called indicator.

no player can be found that beats all of them

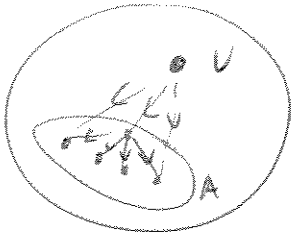
Let $X := \sum_{\text{all } k\text{-subsets } A} X_A = \# \text{ "bad" sets } A$

$$E(X) = \sum_A E(X_A) = \sum_A (1 \cdot \Pr(X_A=1) + 0 \cdot \Pr(X_A=0))$$

$$= \sum_A \Pr(X_A=1) = \binom{n}{k} \cdot \left(1 - \frac{1}{2^k}\right)^{n-k}, \text{ since:}$$

$\Pr(X_A=1) = \Pr(A \text{ is "bad", no } v_A \text{ that beats all others})$

Look at a given set A:



As the probability that there is one to beat all others is $1/2^k$, the probability for a given vertex to be bad is $1 - \frac{1}{2^k}$.
So: $\# \text{ vertices in } S/A$.

$$\prod_{\substack{v \in S \\ v \notin A}} \Pr(v \text{ is "bad" for } A) = \left(1 - \frac{1}{2^k}\right)^{n-k} \quad \square$$

Applying Stirling:

$$\binom{n}{k} \left(1 - \frac{1}{2^k}\right)^{n-k} \leq \left(\frac{en}{k}\right)^k e^{-\frac{n-k}{2^k}} < 1$$

$$1 - x < e^{-x}$$

If the average number of a random variable is less than 1, then there is at least one random variable X that is $\leq E(X)$, so one $X=0$

$\Rightarrow T$ has property S_k . \square

Alternatively:

combi 07/04 6

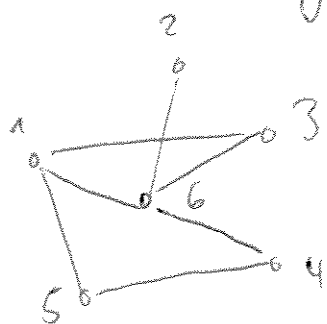
$$\Pr[(A_1 \text{ is bad}) \cup (A_2 \text{ is bad}) \cup \dots \cup (A_k \text{ is bad})] \leq \sum_{i=1}^k \underbrace{\Pr(A_i \text{ is bad})}_{\left(1 - \frac{1}{2^k}\right)^{n-k}}$$
$$= \binom{n}{k} \left(1 - \frac{1}{2^k}\right)^{n-k} < 1$$

$\Rightarrow \Pr(\text{every } A_i \text{ is good}) > 0$, so at least in one random graph, S_k must hold.

So we proved the proposition starting from a random tournament. We have no idea how the tournament looks like, i.e. a purely existential proof. So when you randomly construct graphs satisfy a property but delivering a concrete example is far more difficult.

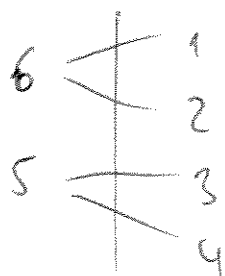
More examples of the probabilistic method:

Problem 1: Finding a large cut in a graph.



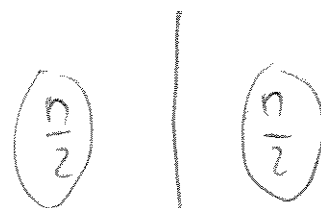
Def.: A cut in a graph is a partition of $V(G)$ into two sets. The size of a cut is # edges going across the two sets.

E.g. there are 2^{n-1} cuts possible on a graph with n vertices.



edges = size of cut

Or consider a complete graph:



$$\Rightarrow \left(\frac{n}{2}\right)^2 \text{ out of } \frac{n^2}{2} - \frac{n}{2}$$

Thm.: Any graph on n vertices and m edges has a cut of size at least $\frac{m}{2}$.

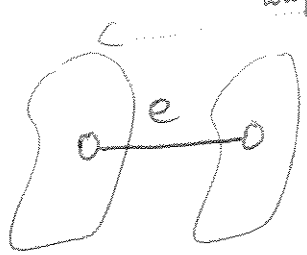
Proof: Key idea: Let's show that there are many cuts that satisfy the statement. Why could that be true and could even be easier than showing that there is one such cut? What's the average size of a cut? If this is large, at least one must be larger than it.

Average size of a cut:
$$\frac{\sum_{\text{all cuts } i} (\text{size of cut } i)}{2^{n-1}}$$

Combi 14/04 2

Double counting: Rather than counting cut by cut, count edge by edge:

$$\frac{\sum_{\text{all cuts } i} (\text{size of cut } i)}{2^{n-1}} = \frac{\sum_{\text{all edges } m \in E} (\# \text{ cuts } m \text{ which it lies)}{2^{n-1}} =$$


$$= \frac{\sum_{e \in E} 2^{n-2}}{2^{n-1}} = \frac{m}{2} \text{ as } |E| = m.$$

So the average is $\frac{m}{2} \Rightarrow$ at least one ^{cut} must be larger than $\frac{m}{2}$.

Alternatively: Probabilistic method.

Let $X :=$ size of a randomly picked cut.

$$E(X) = \sum_{e \in E} E(X_e) = \sum_{e \in E} \Pr(X_e = 1), \text{ where}$$

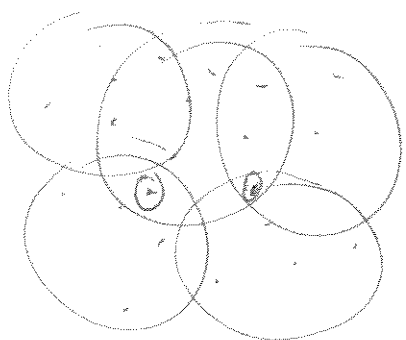
$$X_e := \begin{cases} 1 & \text{edge } e \text{ size of cut } X \\ 0 & \text{if edge } \notin \text{ size of cut } X \end{cases}$$

So $\sum_e \frac{1}{2} = \frac{m}{2} = E(X)$. So again there must

be some cut whose size is at least $\frac{m}{2}$. \square

Problem 2: ϵ -nets

Let the points be people, living in different regions.



Goal: Pick the minimum number of people such that from each region we have someone ~~from each region~~. A representative.

Thm.: Let S_1, \dots, S_m be subsets of X , $|X| = n$, and let X' be the smallest subset of X s.t. $X' \cap S_i \neq \emptyset \forall i$. Assume further that $|S_i| \geq \frac{n}{k}$ for some k and all i .
 $\exists X'$ s.t. $|X'| \leq 2k \ln m$.

Proof: First via double-counting then probabilistic.
 There are $\binom{n}{t}$ subsets of X of size $t := 2k \ln m$.
 Let's count the subsets that do not work. To show:

$$\sum_i (\# \text{subsets without an element of } S_i) = \text{then we are done!}$$

$$= \sum_i \binom{n - |S_i|}{t} < \sum_i \binom{n - \frac{n}{k}}{t} = m \binom{n - \frac{n}{k}}{t} < \binom{n}{t}$$

This inequality holds for $t = 2k \ln m$. (Proof omitted).

Combi 14/04 4

Alternatively: Probabilistic proof.

We want to pick a random set X' and want to show that it fails with probability less than one.

Pick a random set of size t where repetition of the elements is possible. i.e. $|X'| = t$ which we want to pick such that it fails: $\Pr(X' \text{ fails})$.

Let $\mathcal{E}_i :=$ event that set S_i is not "hit" by X' , i.e. $S_i \cap X' = \emptyset$. So

$$\Pr(X' \text{ fails}) = \Pr\left(\bigcup_{i=1}^m \mathcal{E}_i\right) \leq \sum_{i=1}^m \Pr(\mathcal{E}_i) \quad \text{where} \quad \begin{array}{l} \text{union bound} \\ \downarrow \end{array}$$

$$\Pr(\mathcal{E}_i) = \left(1 - \frac{|S_i|}{n}\right)^t. \quad \text{Now let's use another inequality:}$$

$$\sum_{i=1}^m \left(1 - \frac{|S_i|}{n}\right)^t \leq \sum_{i=1}^m \left(1 - \frac{n/k}{n}\right)^t = \sum_{i=1}^m \left(1 - \frac{1}{k}\right)^t =$$

$$m \left(1 - \frac{1}{k}\right)^t \leq m e^{-\frac{t}{k}} = \quad \left(1 - \frac{1}{k} < e^{-\frac{1}{k}} \text{ since } 1+x \leq e^x \forall x\right)$$

$$= m e^{-\frac{t}{k}} = m^{-1} < 1. \quad \text{So } \Pr(X' \text{ fails}) < 1$$

\Rightarrow there must be an X' satisfying the claim. \square

Alternatively, pigeon-hole principle:

Assume an element is in m sets. So

$$m \cdot \frac{n}{k} \leq nr$$

Probabilistic Method

Recall the ε -net problem of last time:

$$X = \{1, \dots, n\}, \quad S_i \subseteq X \text{ s.t. } i \in \{1, \dots, m\} \text{ and } |S_i| \geq \frac{n}{k} \quad \forall i. \quad \exists \tilde{X} \subseteq X \text{ s.t. } \tilde{X} \cap S_i \neq \emptyset \quad \forall i$$

and $|\tilde{X}| = 2k \ln m.$

We saw three ways of proving that, Combinatorial, probabilistic and the "greedy" method from exercise sheet 8. The heart of the probabilistic proof was: Randomly picked t elements:

$$\Pr(\text{at least one set not hit}) \leq m \left(1 - \frac{1}{k}\right)^t$$

let's generalise the problem a bit:

Require $\tilde{X} \cap S_i =$ "at least half the elements of S_i ".

Pick a random sample with choosing an element with probability $\frac{1}{2}$.

Let us allow some error first: 

$\Pr(\# \text{ elements picked from } S_i \text{ are } \frac{|S_i|}{2} \pm t) = \text{not easy, as it is a sum of binomials.}$

Today we will look at tools to evaluate this, namely inequality of Chebychev and inequality of Markov.

combi 21/04 2

let $X = \sum_{j \in S_i} x_j$ where $x_j = \begin{cases} 1 & \text{if } j^{\text{th}} \text{ element is picked} \\ 0 & \text{else} \end{cases}$

$E(X) = \frac{|S_i|}{2}$. What is $\Pr(X \text{ is far from } E(X))$?

(*) Say, $\Pr(X \geq k E(X))$.

E.g.: let there be n elements and pick each with probability $1/2$ and define $x_i = 1$ if i^{th} is picked, zero otherwise.

$$\# \text{ elements picked} = X = \sum_{i=1}^n x_i$$

$$\text{Recall: } E(X) = \sum_{\text{all } X} X \cdot \Pr(X)$$

In a combinatorial setting, consider the unweighted average of n numbers

$$A = \frac{\sum_{i=1}^n a_i}{n}$$

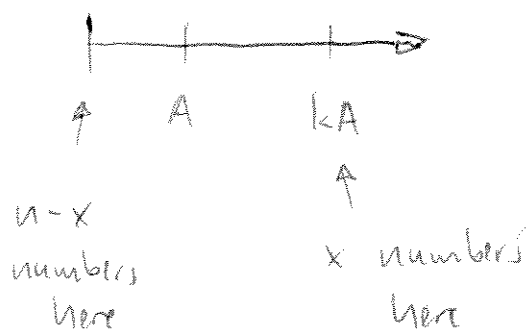
So the equivalent question for (*)

would be: How many a_i 's are $\geq k \cdot A$?

Pick n of the a_i 's such that most are as big as possible,

$$A = \frac{x \cdot kA + (n-x) \cdot 0}{n}$$

$$x = \frac{n}{k}$$



combi 21/04 3

$$A = \frac{1}{n} \sum_{i=1}^n a_i = \frac{1}{n} \left(\sum_{a_i < kA} a_i + \sum_{a_i \geq kA} a_i \right)$$

$$\geq \frac{1}{n} \sum_{a_i \geq kA} a_i \geq \frac{1}{n} \sum_{a_i \geq kA} kA \geq \frac{1}{n} \underline{kA \cdot x} \quad \text{with}$$

$$x \leq \frac{n}{k}. \quad \text{Hence} \quad A \geq \frac{k}{n} \cdot A \cdot x.$$

One can do the same calculation with $E(X)$ instead of A to get Markov's inequality:

$$\Pr(X \geq k \cdot E(X)) \leq \frac{1}{k}$$

So let's do it: let $\Pr(X) := \Pr(X = x)$

$$E(X) = \sum_{\text{all } X} x \Pr(X) = \sum_{X < kE(X)} x \Pr(X) + \sum_{X \geq kE(X)} x \Pr(X)$$

$$\geq \sum_{X \geq kE(X)} kE(X) \Pr(X) = kE(X) \sum_{X \geq kE(X)} \Pr(X),$$

$$\text{so} \quad E(X) \geq kE(X) \Pr(X \geq kE(X))$$

$$\Rightarrow \Pr(X \geq kE(X)) \leq \frac{1}{k}.$$

Note, this is only true for $X \geq 0 \forall X$.

Combi 21/04 4

let's return to the initial setting.

$\Pr(X \geq n) = \frac{1}{2^n}$, but Markov says

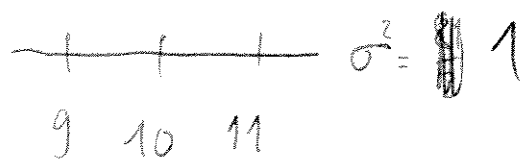
$\Pr(X \geq 2 \cdot \frac{n}{2} = 2 \cdot E(X)) \leq \frac{1}{2}$, i.e. useless here.

Only two assumptions were important for Markov's inequality: $X \geq 0 \forall X$ and $\exists E(X)$.

In order to improve Markov's inequality we must introduce the variance. For example in the following two examples we have the same result of Markov:

$$\Pr(9) = \Pr(11) = \frac{1}{2}$$

$$\Pr(0) = \Pr(20) = \frac{1}{2}$$



i.e. some measure of how far the random variables are away from the mean, $E(X)$, needs to be introduced:

\Rightarrow variance.

$$\text{Var } X := E((X - E(X))^2) = \sigma^2$$

E.g.: $X_i = \begin{cases} 1 \\ 0 \end{cases}$ with prob. $\frac{1}{2}$. $\text{Var } X_i = \frac{1}{4}$.

$$X = \sum_{i=0}^n X_i \Rightarrow E(X) = \frac{n}{2}, \text{ so}$$

$$\text{Var } X = \langle (X - \frac{n}{2})^2 \rangle = \sum_{i=0}^n \left(\frac{n}{2} - i\right)^2 \frac{\binom{n}{i}}{2^n} =$$

$$= \frac{1}{2^n} \left(\sum_{i=0}^n \left(\left(\frac{n}{2}\right)^2 \binom{n}{i} - 2i \frac{n}{2} \binom{n}{i} + i^2 \binom{n}{i} \right) \right) = \frac{n}{4}.$$

Note that $\sum_{i=0}^n i \binom{n}{i} = n 2^{n-1}$

like picking a team of size i and a captain from it.

and $\sum_{i=0}^n i^2 \binom{n}{i} = n 2^{n-1} + (n-1)n 2^{n-2}$ and $\sum_{i=0}^n \binom{n}{i} = 2^n$.

$\Rightarrow \text{Var } X = \frac{n}{4}$. let $\text{Var } X := V[X]$.

Proposition: $V[X+Y] = V[X] + V[Y]$, if X, Y independent.

Proof: By definition: $V[(X+Y)] = E[(X+Y - E(X) - E(Y))^2]$

$$= V[X] + V[Y] - E[2(X - E(X))(Y - E(Y))]$$

(if X and Y are independent, then

$$E(X \cdot Y) = E(X) \cdot E(Y) \quad \square$$

let's motivate Chebychev's inequality by the following: $\Pr((X - E(X))^2 \geq t)$ on which we apply Markov's inequality.

$$\Pr((X - E(X))^2 \geq t) \leq \frac{V[X]}{t} \text{ by Markov.}$$

So $\Pr(|X - E(X)| \geq t) \leq \frac{V[X]}{t^2}$, which is Chebychev's inequality. \square

let us revisit $\Pr(X \geq n) \leq \Pr(|X - \frac{n}{2}| \geq \frac{n}{2})$
 $\leq \frac{\frac{n}{4}}{\frac{n^2}{4}} = \frac{1}{n}$. Still far from $\frac{1}{2^n}$ but better than $\frac{1}{2}$.

E.g.: let's consider the central binomial coefficient $\binom{n}{\frac{n}{2}}$.

As $\sum_{i=0}^n \binom{n}{i} = 2^n$, $\binom{n}{\frac{n}{2}} \geq \frac{2^n}{n+1}$.

We can get a much better bound, by the following argument: let there be n bits, placing 0 or 1 by probability $1/2$. let $X = \# \text{ ones} = \sum_{i=1}^n x_i$

$$X = \sum_{i=1}^n x_i \Rightarrow \Pr(\# \text{ ones are } \frac{n}{2} \pm \sqrt{\frac{n}{2}}) = \sum_{i=\frac{n}{2}-\sqrt{\frac{n}{2}}}^{\frac{n}{2}+\sqrt{\frac{n}{2}}} \frac{\binom{n}{i}}{2^n} \stackrel{1-\frac{1}{2}}{\downarrow} \geq \frac{1}{2}$$

$$\leq \Pr(|X - \frac{n}{2}| \geq \sqrt{\frac{n}{2}}) \leq \frac{\frac{n}{4}}{\frac{n}{2}} = \frac{1}{2}$$

$$\sum_{i=\frac{n}{2}-\sqrt{\frac{n}{2}}}^{\frac{n}{2}+\sqrt{\frac{n}{2}}} \binom{n}{i} \leq 2^n \sum_{i=\frac{n}{2}}^{\frac{n}{2}+\sqrt{\frac{n}{2}}} \binom{n}{i} = 2^{1-n} \sqrt{\frac{n}{2}} \binom{n}{\frac{n}{2}}, \text{ so } \binom{n}{\frac{n}{2}} \geq \frac{2^n}{4\sqrt{\frac{n}{2}}} = \frac{2^n}{e\sqrt{n}}$$

Proofs are exam relevant.

combi 05/05 1

Chebyshev's Inequality

For the random variable X , $E(X) = \mu$, $\text{Var} X = \sigma^2$, then

$$\Pr(|X - \mu| \geq 2\sigma) \leq \frac{1}{2^2} \quad \text{E.g.}$$


Def.: Given A set $\{x_1, x_2, \dots, x_k\}$ is said to be a distinct sum set (DS sum set) if every subset has a distinct sum. i.e. $\forall S_1, S_2 \subseteq \{x_1, \dots, x_k\}$

$$S_1 \neq S_2 \Rightarrow \sum_{x \in S_1} x \neq \sum_{y \in S_2} y.$$

E.g.: $\{1, 2\}$ is a DS set.

Problem: Given $\{1, 2, \dots, n\}$, what is the size of the largest subset that is a distinct sum set?

Clearly $\{2^0, 2^1, 2^2, \dots, 2^i\} \subseteq \{1, 2, \dots, n\}$ as each integer in this subset has a unique binary representation. Hence

$$\lfloor \log_2 n \rfloor \leq \text{"size of largest DS set"} = |\text{DS}|$$

Assume $\{x_1, x_2, \dots, x_k\} = \text{DS}$. Now

$$2^k = \# \text{ of distinct sums} \stackrel{?}{\geq} nk$$

↑
by def
of DS set

$$\Rightarrow k \geq \log_2 nk \geq \log_2 n + \log_2 k$$

combi 05/05 2

$k \leq \log_2 n + \log_2 \log_2 n + 1$ is thus an upper bound of $|DS|$:

$$\lfloor \log_2 n \rfloor \leq |DS| \leq \log_2 n + \log_2(\log_2 n) + 1$$

Thm.: The largest DS subset of $\{1, \dots, n\}$ has size at most $\log_2 n + \frac{1}{2} \log_2 \log_2 n + C$.

Proof: Suppose $\{x_1, x_2, \dots, x_k\} = DS$ i.e. the largest DS subset of $\{1, \dots, n\}$. Take a random sample of DS by picking each element with prob. $\frac{1}{2}$, and then consider the sum.

$$\text{let } \varepsilon_i = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ } x \text{ is picked} \\ 0 & \text{else} \end{cases}$$

$$E(\varepsilon_i) = \frac{1}{2} \quad \text{and} \quad \text{Var } \varepsilon_i = 1 \cdot \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

let $X = \varepsilon_1 x_1 + \varepsilon_2 x_2 + \dots + \varepsilon_k x_k$, i.e. the sum of the random subset.

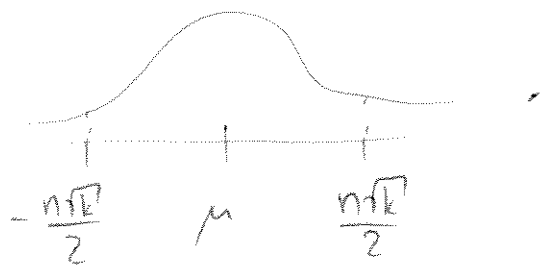
$$E(X) = \sum_{i=1}^k E(\varepsilon_i x_i) = \sum_{i=1}^k x_i E(\varepsilon_i) = \frac{1}{2} \sum_{i=1}^k x_i$$

$$\text{Var } X = \sum_{i=1}^k \text{Var}(\varepsilon_i x_i) = \sum_{i=1}^k x_i^2 \text{Var } \varepsilon_i = \frac{1}{4} \sum_{i=1}^k x_i^2$$

By Chebyshev: $\Pr(|X - \mu| > 2\sigma) < \frac{1}{2^2}$

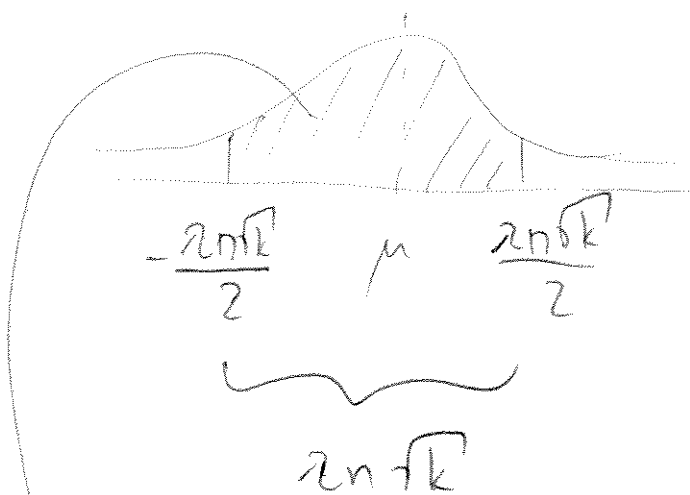
The variance can easily bound above by $\text{Var } X \leq \frac{n^2 k}{4}$,

so $\sigma \leq \frac{n\sqrt{k}}{2}$. E.g.



Hence $\Pr\left(|X - \mu| > \frac{n\sqrt{k}}{2}\right) < \frac{1}{2^2}$

$\Rightarrow \Pr\left(|X - \mu| \leq \frac{n\sqrt{k}}{2}\right) \geq 1 - \frac{1}{2^2}$.



$\{x_1, x_2, \dots, x_k\}$

↓ random sample
and add terms:
there must be 2^k distinct
sums, by def. of DS,

picture



$$\leq \frac{2n\sqrt{k}}{2^k}$$

$\Pr(X \text{ is in the shaded region})$

$\Pr\left(|X - \mu| < \frac{2n\sqrt{k}}{2}\right)$. So

$$1 - \frac{1}{2^2} \leq \Pr\left(|X - \mu| < \frac{2n\sqrt{k}}{2}\right) \leq \frac{2n\sqrt{k}}{2^k}$$

Pick a ~~the~~ some α , say $\alpha = \frac{1}{3}$ to get

$$2^k \left(1 - \frac{1}{3}\right) \leq \frac{2\sqrt{3}}{2} n\sqrt{k} \Rightarrow 2^k \leq \frac{3\sqrt{3} n\sqrt{k}}{2}$$

Hence $k \leq \log_2 n + \log_2 \sqrt{k} + \log_2 \frac{3\sqrt{3}}{2} \leq \log_2 n + \frac{\log_2 \log_2 n}{2} + c$

□

More probabilistic methods:

Def.: $G(V, E)$ is a planar graph if it can be drawn in the plane without crossing edges.

G is planar iff G does not have a K_5 or $K_{3,3}$ topological minor.

Thm.: If G is planar and $V(G) = n$, then $|E(G)| \leq 3n - 6$.

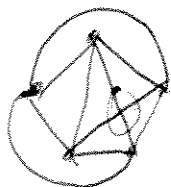
Proof: $\sum_{f \in F(G)} d(f) = 2|E(G)| \geq \sum_{f \in F(G)} 3 = 3f$, i.e.

$2m \geq 3f$ and by Euler, $n - m + f = 2$,

so $m \leq 3n - 6$. □

Def.: The crossing number of G is the minimum number of crossing edges over all drawings of G . (i.e. the "best" drawing of G). Let the crossing number of G be $cr(G)$.

E.g.: $cr(K_4) = 0$, $cr(K_5) = 1$



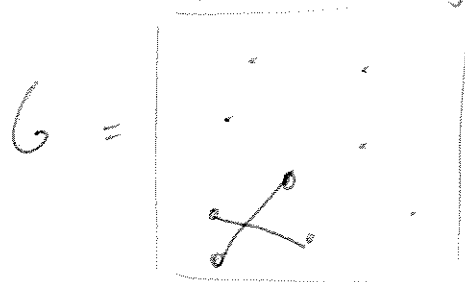
combi 05/05 5 exam relevant

Lemma: (Crossing lemma) If G is a n -vertex graph with $e \geq 4n$, then $(|E(G)| = e)$

$$cr(G) \geq \frac{e^3}{64n^2}.$$

Proof: Clearly, $cr(G) \geq e - (3n - 6) = e - 3n + 6 \geq e - 3n$
So $cr(G) \geq e - 3n$ for any graph.

Let us pick each vertex of G with probability p , and require that both vertices of an edge must be picked for the edge to survive.



Let $X := cr(G') - e' + 3n' \geq 0$
where G' is picked from the given graph. So it is a random variable.

$$E(X) = E(cr(G')) - E(e') + 3E(n') \geq 0.$$

Let the picked G' be drawn with minimal crossings. A given crossing in G^m survives if 4 vertices are chosen, hence $E(cr(G')) = cr(G) \cdot p^4$. Thus

$$E(X) = cr(G)p^4 - ep^2 + 3np \geq 0$$

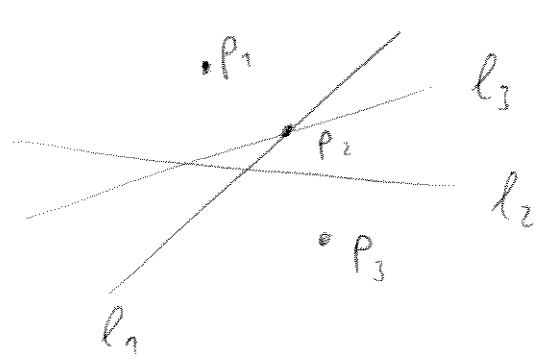
$\Rightarrow cr(G) \geq ep^{-2} - 3np^{-3} = f(p)$ and maximize

$f(p)$, so $f'(p) = 0 \Rightarrow p = \frac{4n}{e} \leq 1$ so a valid probability. Plugging this in gives: $cr(G) \geq \frac{e^3}{64n^2}$. \square

Combi 05/05 6

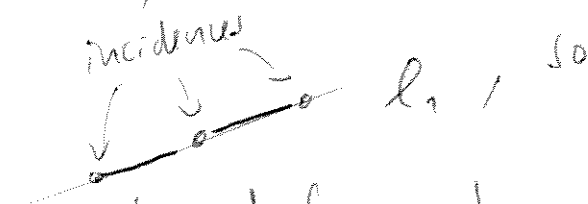
Problem: Let there be n lines and n points in \mathbb{R}^2 , what is the maximum # incidences between points and lines? E.g.

Two incidences:
 $(P_2, l_3); (P_2, l_1)$.



Thm.: Given n points and n lines,
$$\max \# \text{ incidences} \leq C \cdot (n^{4/3} + n)$$

Proof: Define a graph G as follows. Let $V(G)$ be the n points and $E(G)$ be two consecutive points on a line. E.g.



which we want to bound from above,

$$\# \text{ incidences} \leq |E(G)| + \frac{\# \text{ lines}}{n} = |E(G)| + n$$

By the crossing lemma,

$$cr(G) \geq \frac{|E(G)|^3}{64 n^2}$$

A crossing can happen when 2 lines cross, so the maximum number of crossings is $\binom{n}{2}$, so

$$\frac{|E(G)|^3}{64 n^2} \leq cr(G) \leq \binom{n}{2} \leq \frac{n^2}{2}$$

Combi 05/05 7

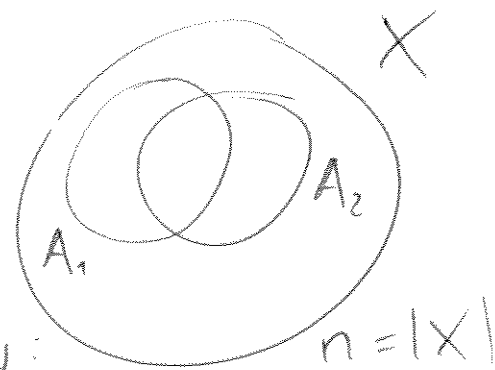
$$|E(G)|^3 \leq 32 n^4 \Rightarrow |E(G)| \leq \sqrt[3]{32} n^{4/3}$$

$$\# \text{ incidences} \leq \sqrt[3]{32} \cdot n^{4/3} + n \quad \square$$

Linear Algebra Method

Introductory problems:

1) Let there be n people in a city who form clubs according to two rules:



a) $|A_i| = \text{even} \quad \forall i$

b) $|A_i \cap A_j| = \text{even} \quad \forall i \neq j$

What is the max # sets A_i that satisfy these rules?

E.g. clearly satisfies the conditions. So at least $\frac{n}{2}$ sets can be chosen.

See each point couple as one point to see that there are at least $2^{n/2}$ clubs.

Hw: Is this the best possible lower bound.

2) New rules:

a) $|A_i| = \text{odd} \quad \forall i$

b) $|A_i \cap A_j| = \text{even} \quad \forall i \neq j$

Same question: What is the maximum number of A_i satisfying those rules?

Select a special point and include it in each club, but here rule b) doesn't work.

Thm.: (For 2) max # sets is n

Proof:

$$\begin{array}{c}
 A_1 \\
 A_2 \\
 \vdots \\
 A_m
 \end{array}
 \begin{pmatrix}
 x_1 & x_2 & \dots & x_n \\
 0 & 1 & \dots & 1 \\
 1 & 0 & \dots & 0 \\
 \vdots & \vdots & & \vdots \\
 \vdots & \vdots & & \vdots \\
 \vdots & \vdots & & \vdots
 \end{pmatrix}$$

Consider the following incidence matrix.

If x_j is in A_i then the ij entry is one, else zero. (call this matrix (a_{ij})).

For the sake of contradiction, assume $m > n$.

Claim: One can assign constant λ_i to sets A_i such that $\lambda_i \in \{0, 1\}$, letting 0 represent even and 1 represent odd, such that

$$\sum_{i=1}^m \lambda_i v_i = (0, 0, \dots, 0), \text{ where } v_i \text{ is the}$$

i th row vector, and not all λ_i 's are zero.

(# choices for $(\lambda_1, \dots, \lambda_m) = 2^m$ and for each choice $\sum_{i=1}^m \lambda_i v_i = (\mu_1, \mu_2, \dots, \mu_n) \in \{0, 1\}^n$ where each μ_j is either odd or even (1 or 0).

If $2^m > 2^n$ then there will be two choices

$(\lambda_1', \dots, \lambda_m')$ and $(\lambda_1'', \dots, \lambda_m'')$ such that

$$\sum_{i=1}^m \lambda_i' v_i \equiv \sum_{i=1}^m \lambda_i'' v_i \pmod{2}$$

$$\sum_{i=1}^m (\lambda_i' - \lambda_i'') v_i \equiv \underline{0} \pmod{2}. \text{ End of claim.}$$

One can also see this using this fact from linear algebra
Any set of linearly independent vectors in an n -dimensional vector space has at most n elements.

Let $\alpha_i' - \alpha_i'' = \alpha_i$, then we found α_i s.t.

$$\sum_{i=1}^m \alpha_i v_i \equiv \underline{0} \pmod{2}, \text{ take dot product with } v_1:$$

$$\sum_{i=1}^m \alpha_i \langle v_i, v_1 \rangle \equiv \underline{0} \pmod{2}$$

even by rule b)
even by rule b)

$$\alpha_1 \underbrace{\langle v_1, v_1 \rangle}_{\text{odd by rule a)}} + \alpha_2 \langle v_2, v_1 \rangle + \dots + \alpha_m \langle v_m, v_1 \rangle \equiv \underline{0}$$

$$\Rightarrow \alpha_1 \equiv \underline{0}$$

Next multiply (dot product) with $v_2 \Rightarrow \alpha_2 = 0$. This can be done with all v_i , thus $\alpha_i \equiv 0 \forall i$. But this contradicts our assumption. ~~X~~

Thm. i (Fisher's Inequality)

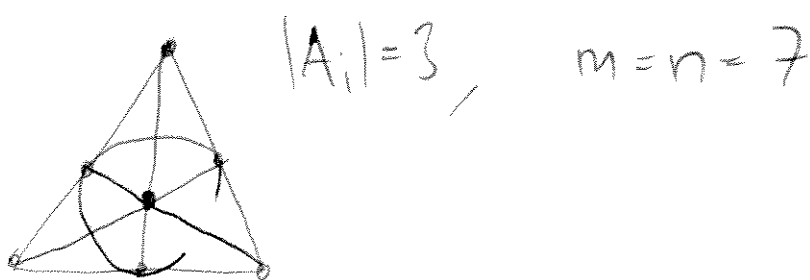
Let $A_1, \dots, A_m \subseteq X$, $|X| = n$ and $|A_i \cap A_j| = k \forall i \neq j$.

Then $m \leq n$.

E.g. : i) $k=1$,



ii) Fano Plane, 7 pts, 7 sets A_i ("lines"), s.t. any 2 pts can be connected by precisely one line and any 2 lines have precisely one pt in common.



Proof: Again an incidence matrix: $\#$ m row vectors v_i :

$x_1 \ x_2 \ \dots \ x_n$

$A_i \rightarrow v_i = (0, 1, \dots, 1) \in \{0, 1\}^n$, so $\langle v_i, v_i \rangle = |A_i|$.

For the sake of contradiction, assume that $m > n$, then

$\exists \sum_{i=1}^m \lambda_i$, $i \in \{1, \dots, m\}$, not all zero, such that

$\sum_{i=1}^m \lambda_i v_i = \underline{0} \in \mathbb{R}^n$, i.e. there must be a non-trivial

linear dependence. Now consider $\langle \sum_{i=1}^m \lambda_i v_i, \sum_{j=1}^m \lambda_j v_j \rangle$

$$= \langle \underline{0}, \underline{0} \rangle = \sum_{i=1}^m \lambda_i^2 \underbrace{\langle v_i, v_i \rangle}_{|A_i|} + \sum_{i \neq j} \lambda_i \lambda_j \underbrace{\langle v_i, v_j \rangle}_{|A_i \cap A_j|} =$$

Remark: $|A_i| > k \ \forall i$, except perhaps 1.

combi 12/05 5

$$\begin{aligned} &= \sum_{i=1}^m \lambda_i^2 (|A_i| - k) + \sum_{i \neq j} \lambda_i \lambda_j k + \sum_{i=1}^m \lambda_i^2 k = \\ &= \sum_{i=1}^m \lambda_i^2 (|A_i| - k) + k \cdot \left(\sum_{i=1}^m \lambda_i \sum_{j=1}^m \lambda_j \right) = \underline{0} \end{aligned}$$

$\underbrace{\phantom{\sum_{i=1}^m \lambda_i^2 (|A_i| - k)}}_{> 0 \forall i}$

A sum of positive terms equals to zero if both summands are zero. This only holds if $\lambda_i = 0 \forall i$. ~~X~~

combi: 19/05 1

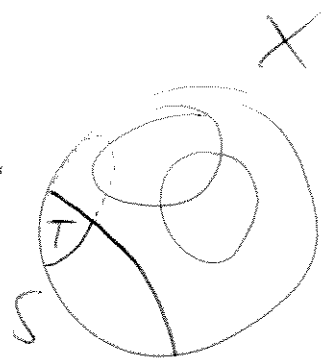
More about the linear algebra method.

Thm.: (Sauer, Shelah, Vapnik - Chervonenkis)

Let $A_1, A_2, \dots, A_m \subseteq X$ where $|X| = n$.

Call $S \subseteq X$ shattered iff $\forall T \subseteq S$

$\exists A_i$ s.t. $A_i \cap S = T$. Then:



i) If there is a shattered set of k elements, then
 $m \geq 2^k$.

ii) If we take all subsets of X that have at most k elements then there is no shattered set of size $k+1$.

If $m > \sum_{i=0}^k \binom{n}{i}$ then there is a shattered set of size $\geq k+1$. This bound is tight (see ii)).

Recall: "Linear algebra method": In every n -dimensional vector space, the # linearly independent vectors is at most n .

Proof:

It's contrapositive: If there is no shattered set of size $k+1$

$$\text{then } m \leq \underbrace{\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{k}}_{\text{dimension of space.}}$$

we cannot ~~disprove~~
prove strict inequalities via
the linear algebra method

Combi 19/05 2

Let us define an incidence matrix:

$$\begin{matrix}
 & \binom{n}{0} & \binom{n}{1} & \dots & \binom{n}{k} \\
 & \emptyset & \{x_1\}, \{x_2\}, \dots, \{x_n\} \\
 A_1 & 1 & 1 & 1 & 1 & \dots \\
 A_2 & 1 & 0 & 1 & 1 & \dots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 A_m & 1 & 0 & 1 & 0 & \dots
 \end{matrix}
 \quad a_{ij} = \begin{cases} 1 & \text{if } B_j \subseteq A_i \\ 0 & \text{else} \end{cases}$$

For each k -element subset of X there is a column.

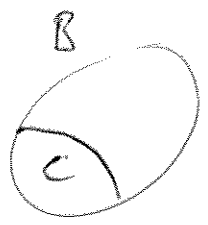
Let us consider the column vectors now. The ones denote the # A_i that contain a given B_j .

Suppose for contradiction that $m > \binom{n}{0} + \dots + \binom{n}{k}$, i.e. more rows than columns. A row vector v_i is element of $\{0, 1\}^{\sum_{j=0}^k \binom{n}{j}}$. The assumption implies $\exists \alpha_i$, not all zero, s.t. $\sum_{i=1}^m \alpha_i v_i = 0 \in \{0, 1\}^{\sum_{j=0}^k \binom{n}{j}}$, i.e. adding up each column (multiplied by α_i row-wise) must be zero. So for every at most

k -element subset $B_j \subseteq X$, $\sum \alpha_i = 0$. We can define this sum for any set. $B_j \subseteq A_i$ So let $B \subseteq X$ and define $f(B) := \sum_{A_i \supseteq B} \alpha_i$. Consider a smallest set $B \subseteq X$

for which $f(B) = \alpha \neq 0$. One such set is the A_i that contains the largest number of elements and for which $\alpha_i \neq 0$: $f(A_j) = \sum_{A_i \supseteq A_j} \alpha_i = \alpha_j \Rightarrow |A_j| \geq k+1$.

combi 19/05 3



$f(B) = \alpha \neq 0, |B| \geq k+1$

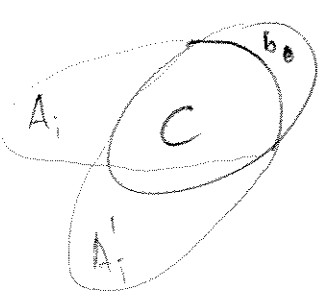
claim: $\forall C \subseteq B, \overline{f}_B(C) = (-1)^{|B \setminus C|} \cdot \alpha \neq 0$

This holds, since

$\overline{f}_B(C) := \sum_{B \cap A_i = C} \alpha_i$ and note that $\overline{f}_B(B) = f(B)$.

So $\overline{f}_B(C) = \sum_{A_i \cap B = C} \alpha_i \neq 0 \Rightarrow \exists A_i \cap B = C \Rightarrow B$ is shattered.

This follows from the inclusion-exclusion principle: For instance if $C = B \setminus \{b\}$, consider



$$\overline{f}_B(C) = \sum_{A_i \cap B = C} \alpha_i = \sum_{A_i \supseteq C} \alpha_i - \sum_{A_i \supseteq B} \alpha_i =$$

$$= f(C) - f(B) = f(C) - \alpha$$

$$= -\alpha$$
 since we defined

B as the smallest set for which $f(B) \neq 0$, thus $f(C) = 0$.

This can be generalised via induction: E.g.

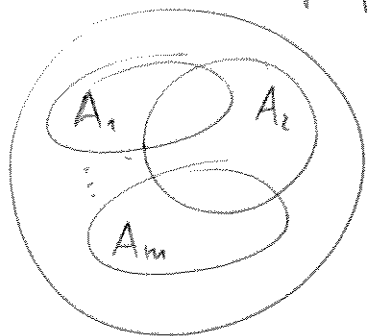


$\Rightarrow -\alpha + \alpha + \alpha$

□

Combi 19/05 - 4

Application: n people



$$|A_i| \geq \epsilon n \quad (i = 1, \dots, m)$$

Pick a $\#$ people s.t. $\forall A_i$ is "fairly" represented.

If there is no shattered set of size $k+1$ then

$C_k \frac{1}{\epsilon^2} \log \frac{1}{\epsilon}$ representatives are sufficient

(Vapnik-Chervonenkis). Note that the $\#$ representatives does not depend on n . So even in a big country one need not ask more people than in a small country. The selection of those representatives is more important than the total number of participants.

Thm.: (Larman-Rodgers-Seidel)

Def.: $X \subseteq \mathbb{R}^d$ is said to be a k -distance set if there are at most k distinct distances $\|x-y\|$ determined by $x \neq y \in X$.

E.g.: 1-distance set in \mathbb{R}^2 :



At most 3 elements.

$$\text{In } \mathbb{R}^d, |S| \leq d+1.$$

If $S \subseteq \mathbb{R}^d$ is a 2-distance set \Rightarrow

$$|S| \leq \frac{(d+1)(d+4)}{2} \quad (\text{"nearly tight"}).$$

combi 19/05 5

Proof: (Linear algebra)

The linear space will be a space of polynomials here.
Let the two distances of S be denoted by α and β .
 $\forall s \in S$, define a function on $t \in \mathbb{R}^d$

$$F_s(t) = (\|t-s\|^2 - \alpha^2)(\|t-s\|^2 - \beta^2)$$

If $t \neq s$ and $t \in S$ then $F_s(t) = 0$, and $F_s(s) = (\alpha\beta)^2 \neq 0$.

There exist no $\lambda_1, \dots, \lambda_{|S|}$, not all zero, such that $\sum_{s \in S} \lambda_s F_s(t) = 0$. I.e. the functions $F_s(t)$

$s \in S$ are linearly independent. This can be seen by setting

$$t = s : \lambda_s \alpha^2 \beta^2 = 0 \Rightarrow \lambda_s = 0 \quad \forall s \in S \quad \downarrow$$

It remains to prove that all functions $F_s(t)$, $s \in S$, belong to a low-dimensional vector space of polynomials:

$$\|t-s\|^2 = \langle t-s, t-s \rangle = \|t\|^2 - 2 \underbrace{\langle s, t \rangle}_{\sum_{i=1}^d s_i t_i} + \|s\|^2$$

So, as $\|t\|^2 = t_1^2 + \dots + t_d^2$. Multiply out $F_s(t) =$

$$\|t\|^4 - 4\|t\|^2 \sum_i s_i t_i + 8 \sum_{i < j} s_i s_j t_i t_j + 4 \sum_i s_i^2 t_i^2 + \dots \rightarrow$$

combi 26/05 1

1. Algorithms ("main customer of combinatorics")
2. Ramsey theory
3. Geometric applications

Prop.: Let there be a family of sets A_1, \dots, A_m , $|A_i| = k$ and $m < 2^{k-1}$. There exists a 2-coloring of $\bigcup_{i=1}^m A_i$ such that no A_i is monochromatic.



Proof: (Erdős)

Consider a random coloring of the points, probs. $\frac{1}{2}$, we want to show: $\Pr(\exists \text{ monochromatic } A_i) > 0$.

$\Pr(\exists \text{ monochromatic } A_i) \leq m \frac{2}{2^k} = \frac{m}{2^{k-1}} < 1$ by assumption $\Rightarrow \exists$ a 2-coloring. \square

Alternatively: Algorithmic proof.

Proof: (Erdős - Selfridge)

Greedy algorithm. (Consider two people alternatively trying to have a monochromatic set by painting a point orange or green ~~blue~~). Strategy for 2 players: Choose the point that is "most important" and color it by orange or green. So how do we define most important?

combi 26/05 2

○ = orange
● = green

$w(A_1) = 2$

1st player (he starts the game):

Each set A_i gets a weight $w(A_i)$.

Originally $w(A_i) = 1 \forall i$. If

1st player puts a point into A_i , then $w(A_i) = 0$, since he no longer has to worry about the set since it can no longer be made monochromatic in green. The more green points in a set, the more dangerous it becomes for the 1st player, thus define a set with green points only as having weight

$$w_1(A_i) = 2^{(\# \text{ green points in } A_i)}$$

The initial condition is:

$$\sum_{i=1}^m w_1(A_i) = m < 2^{k-1}$$

The weight of a point p ,

$$w_1(p) = \sum_{p \in A_i} w_1(A_i)$$

since it is more important if it is in more sets.

Now again: Strategy for the players is such that at each turn the point with the highest weight $w(p)$ is painted.

2nd player (green): $w_2(A_i) = 2^{(\# \text{ orange points in } A_i)}$

$w_2(A_i) = 0$ if A_i has a green pt.

$$w_2(p) = \sum_{A_i \ni p} w_2(A_i)$$

Total weight for 2nd player (after 1st player made his first choice):

$$\sum_{i=1}^m w_2(A_i) \leq 2m < 2^k$$

Essence of the proof: If both players follow the just described greedy strategy then their total weights can only decrease! So neither of the players can win!

Let us look at this from the point of view of the orange player: Green cannot win, since for that a set must have weight 2^k , but this is impossible since

$$\sum_{i=1}^m w_1(A_i) = m < 2^{k-1} \text{ initially and only decreases. The}$$

same is true for the green player, since $\sum w_2(A_i) \leq 2m < 2^k$.

Why is this true that the total weights can only decrease?

From point of view of orange player, consider the last point that orange picked: As it was the best point possible,



$$w_1(p) = \sum_{A_i \ni p} w_1(A_i) \text{ was maximal,}$$

and thus the total weight for orange decreases by $w_1(p)$ since $w_1(A_i) = 0$ if an orange point is in it.

But if green picks a point q , the total weights of A_i increase by $w_1(q)$. But the decrease is bigger than the

Combi 26/05 4

increase, since: $w_+(p) \geq w_+(q)$.

A similar argument shows that the total weight for the green player only decreases. \square

The idea of introducing "fighting players" to replace the probabilistic method is the most popular.

EXAM: Written, 5 or 6 questions,

- 1 proof of a theorem covered in class,
send a list to (janospach@gmail.com)?
 - At least one problem from a homework assignment
(remember solutions)
-

2 Ramsey Theory

Thm.: $\forall n \exists R(n)$ s.t. no matter how we color the edges of a complete graph with $\geq R(n)$ vertices by two colors, there exists a monochromatic complete subgraph K_n .

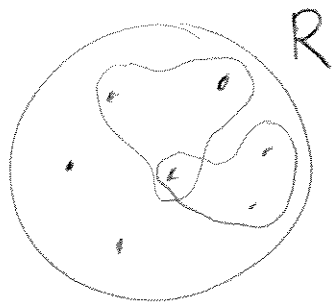
One can extend the problem by introducing more colors; then the minimal number of vertices such that a monochromatic complete graph is guaranteed will depend on c as well: $R(n, c)$.

combi 26/05 5

For hypergraphs, take more vertices to define an edge:

Triples of vertices: 3-uniform hypergraph

Quadruples of " : 4-uniform hypergraph



All $\binom{R}{3}$ triples, i.e. a complete 3-uniform hypergraph on R vertices.

Color each triple by one of c given colors.

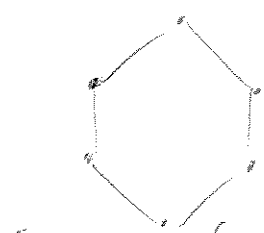
Thm.: (General Ramsey Theorem) 1930 (Happy Ending Theorem)

For every $n, c \exists R_3(n, c)$ such that no matter how we color all triples of an $\geq R_3(n, c)$ -element set X with c colors, there is an n-element subset $Y \subseteq X$ such that all $\binom{Y}{3}$ triples induced by them are of the same color.

Thm.: (Erdős-Szekeres Convex n-gon theorem 1935)

For every n, there exists $E(n)$ such that any set of $\geq E(n)$ points in the plane, no three of them collinear, contains the vertex set of a convex n-gon.

E.g.:



convex 6-gon

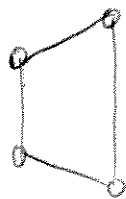
(When spanning a rubber band around, all vertices in a convex graph are touched)

$n=3$

$E(3) = 3$



$n=4$



but

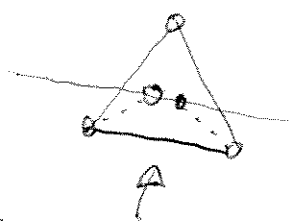


"bad"

$E(4) = 5$,
as Ester-Klein
showed that

with any 5 points in the plane, not three of them collinear, one can place a "rubber band" around them and the four touched vertices are in convex position.

Problematic case:



here is the convex subgraph

Pasch axiom: A line cannot intersect all three sides of a triangle unless it passes through a vertex.

i.e. ES-thm. says that if one has enough points, one can find a convex n -gon.

Proof: $E(n) \leq \underbrace{R_4(n, 2)} = R, \quad n \geq 5,$

if we color all 4-tuples of an R -element set by 2 colors then we find n elements such that all $\binom{n}{4}$ 4-tuples induced by them are of the same color.

Combi 26/05 7



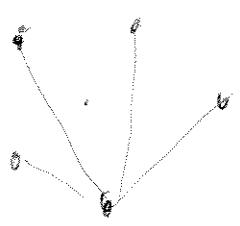
R points, each 4 element subset is colored in one of two colors.

different color, since the convex hull is a triangle.

~~By Ramsey's theorem~~

By E. Klein's observation there are no 5 pts. such that all 4-tuples determined by them are green. So \exists

n pts. such that all 4-tuples determined by them are orange. \Rightarrow These n points form the vertex set of a convex n -gon.



Triangulation \square