Discrete Mathematics 2013 – Problem Set 9

You can hand in a solution of the star problem until Thursday, November 21st, 11:15 a.m.

1. A *complete graph* is a graph with any two vertices connected by an edge. Prove that the edges of an n-vertex complete graph can be colored with two colors so that at most $\binom{n}{k}2^{1-\binom{k}{2}}$ of its complete k-vertex subgraphs are monochromatic.

2. A *tournament* is a directed graph which contains exactly one directed edge connecting any two distinct vertices u and v: either (u, v) or (v, u). A *Hamiltonian path* in a tournament is a directed path that goes through all the vertices, each exactly once (including the starting and ending vertices). Prove that for every $n \in \mathbb{N}$, there is a tournament on n vertices with at least $n!/2^{n-1}$ Hamiltonian paths.

3. Prove that if $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$, then there exists a tournament on n vertices with the following property: for any set S of k vertices there is a vertex $u \notin S$ such that there is a directed edge from u to every vertex in S.

4. Let G be a graph with vertex set V and with m edges. Consider the following procedure finding a partition $V = A_0 \cup A_1$.

   Initially, partition V arbitrarily into two disjoint subsets A_0 and A_1. Then, repeatedly, choose any vertex $v \in A_i$ (with any $i \in \{0, 1\}$) that has more neighbors in A_i than in A_{1-i}, remove it from A_i, and put it into A_{1-i}. Stop when all vertices in A_i have at most as many neighbors in A_i as in A_{1-i}, for $i \in \{0, 1\}$.

   Prove that this procedure always stops and results in a partition $V = A_0 \cup A_1$ such that at least $m/2$ edges of G go between A_0 and A_1.

5. * Let G be a graph with n vertices and m edges. Consider the following procedure finding an independent set I in G.

   Initially, let $I = \emptyset$ and S be the set of all vertices. Then, repeatedly, choose a vertex $v \in S$ whose degree in G is minimum among all vertices in S, add v to I, and remove v and all its neighbors from S. Stop when S becomes empty.

   Prove that this procedure results in an independent set I with

$$|I| \geq \frac{n^2}{2m + n}.$$