Discrete Mathematics 2013 – Problem Set 13

1. Compute the coefficient of power x^{12} in the power series of the function
$$\left(x \cdot \frac{1-x^6}{1-x}\right)^3.$$
Use the result to answer the following question: What is the probability of getting exactly 12 points when rolling 3 dice?

2. Let $\mathcal{F}$ be a family of sets and k be the size of the smallest set in $\mathcal{F}$. Prove that if any $k+1$ of the sets in $\mathcal{F}$ have non-empty intersection, then all the sets in $\mathcal{F}$ have non-empty intersection.

3. Let P be a set of n points in the plane, not all on the same line. Prove that there are at least n distinct lines passing through at least two points in P.

   Hint: For each point in P, consider the set of lines passing through this point, and use Fisher's inequality on these sets.

4. Let $\mathcal{F}$ be a family of subsets of $\{1, 2, \ldots, n\}$ with the following two properties:
   - the size of every set in $\mathcal{F}$ is odd,
   - any two sets in $\mathcal{F}$ have an even number of common elements.

   Let $\mathbb{F}_2$ denote the 2-element field—the set $\{0, 1\}$ with addition and multiplication modulo 2. For every set $A \in \mathcal{F}$, we define its characteristic vector $(a_1, a_2, \ldots, a_n) \in \mathbb{F}_2^n$ so that
$$a_i = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if } i \notin A. \end{cases}$$
   Prove that the characteristic vectors of all the sets in $\mathcal{F}$ are linearly independent vectors in the vector space $\mathbb{F}_2^n$ over the field $\mathbb{F}_2$. Use this fact to deduce that $|\mathcal{F}| \leq n$.

   Hint: Suppose that there is a linear combination of the characterictic vectors equal to zero. Compute its inner product with any of the characteristic vectors.