Note that in the second page, instead of the inequality

$$\Pr[A_{STV}] \leq 2^{-(2n-1)} p^{v},$$

we have

$$\Pr[A_{STV}] \leq 2^{-(2n-1)} p^{\mathbf{v+1}}.$$

$E[Y_e]=p^2$ as for e to be inside S two vertices must be chosen. As

$$Y=\sum_{e\in E} Y_e, \qquad E(Y)=\sum_{e\in E} E(Y_e)=\binom{nk}{2}p^2,$$

then

$$E(X-Y)=E(X)-E(Y)=np-(nk/2)p^2.$$

Let us choose $p=1/k$ to maximize the above expression:

$$E(X-Y)=n/k.$$

There is a point in the probability set for which $X-Y$ is at least n/k. That is, there is a set S which has at least n/k more vertices than edges. Delete one point from each edge from S leaving a set S^*. Then S^* is independent and has at least n/k points.

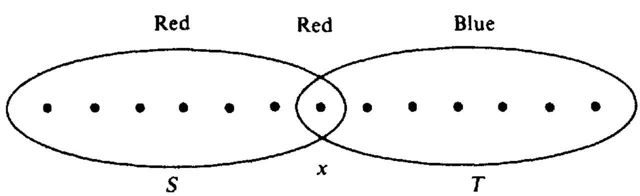
In a formal proof the variable p would not appear; S would be defined by $\Pr[x\in S]=1/k$. The neophyte, reading such a definition in the opening line of a proof in a research paper, might be somewhat mystified as to where the "$1/k$" term came from. Rest assured, this value was a variable when the result was being found. One way to understand a probabilistic proof is to reset all the probabilities to variables and see that the author has, indeed, selected the optimal values.

Property B. Let us return to the Property B function $m(n)$ of Lecture 1. Suppose $m=2^{n-1}k$ and F is a family of m n-sets with Ω the underlying point set. We give a recoloring argument of J. Beck that shows Ω may be two-colored without monochromatic $S\in F$ as long as $k<n^{1/3-o(1)}$.

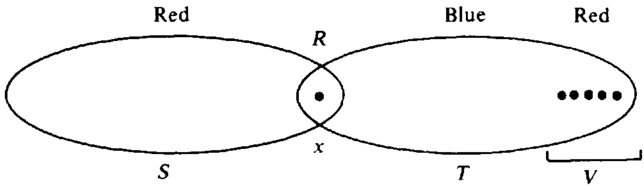
First we color Ω randomly. This gives X monochromatic $S\in F$ where $E(X)=k$. Now we *recolor.* For every $x\in\Omega$ which lies in some monochromatic SF we change the color of x with probability $p=[(\ln k)/n](1+\varepsilon)$. We call these the First and the Second Coloring or Stage. To be precise, even if x lies in many monochromatic $S\in F$ the probability of its color changing is just p.

Given that S was monochromatic at the First Stage the probability that it remained so was $(1-p)^n+p^n\sim(1-p)^n\sim e^{-pn}=k^{-1-\varepsilon}$. The expected number of S monochromatic at both stages is then $k(k^{-1-\varepsilon})=k^{-\varepsilon}$, which is negligible. Of course, p was selected just large enough to destroy the monochromatic S. Has the cure been too strong, so that new monochromatic T have appeared?

For each $S, T\in F$ with $S\cap T\neq\varnothing$ let A_{ST} be the event that S was Red at Stage 1 and T became Blue at Stage 2. That is, in curing S we have created T. We will bound $\Pr[A_{ST}]$ in the case $|S\cap T|=1$, say $S\cap T=\{x\}$. It can be shown (exercise) that $\Pr[A_{ST}]$ is even smaller when $|S\cap T|$ is larger. A_{ST} can occur if, for instance, the following conditions hold:



This occurs with probability $2^{-(2n-1)}p$ since the First Coloring of $S\cup T$ is determined and then x must change color. More generally, though, for $V\subseteq T-S$ with $|V|=v$, let A_{STV} be the following event:



$S\cup V$ is Red at Stage 1, the rest of T is Blue at Stage 2, and $V\cup\{x\}$ changes color at Stage 2. As the color of $S\cup T$ is determined,

$$\Pr[A_{STV}]=2^{-(2n-1)}\Pr[V\cup\{x\}\text{ becomes Blue}\mid\text{First Coloring of } S\cup T].$$

This conditional probability is at most p^{v+1} as each of the $v+1$ points $V\cup\{x\}$ must turn Blue. (Perhaps it is much less than that, as each $y\in V$ must also lie in a Red set. I feel that, if we could take full advantage of this additional criterion, the bound on $m(n)$ could be improved.) Thus

$$\Pr[A_{STV}]\le 2^{-(2n-1)}p^{v}.$$

Hence

$$\begin{aligned}\Pr[A_{ST}]&\le\sum_V\Pr[A_{STV}]\le\sum_{v=0}^{n-1}\binom{n-1}{v}2^{-(2n-1)}p^{v+1}\\&=p2^{1-2n}(1+p)^{n-1}\qquad\text{(Binomial Theorem)}\\&\sim cp2^{-2n}e^{pn}\sim cp2^{-2n}k^{1+\varepsilon}.\end{aligned}$$

(We can calculate that the main contribution to the sum occurs not at $v=0$ but rather near $v=\ln k$. Can we use that these $\ln k$ points all lie in Red sets at Stage 1?) There are at most $(2^{n-1}k)^2=c2^{2n}k^2$ choices of S, T so

$$\begin{aligned}\Pr[\vee A_{ST}]&\le c(2^{2n}k^2)(2^{-2n}k^{1+\varepsilon}p)\\&=ck^{3+\varepsilon}p=ck^{3+\varepsilon}(\ln k)/n\ll 1\end{aligned}$$

if $k=n^{1/3-o(1)}$. That is, for this k the expected number of monochromatic sets at Stage 2 is much less than one so with positive probability there are no monochromatic sets. Thus there exists a coloring of Ω with no monochromatic sets. Thus $m(n)>c2^n n^{1/3-o(1)}$. In fact, with a bit more care, $m(n)>c2^n n^{1/3}$ may be shown.

Erdös has shown the upper bound $m(n)<c2^n n^2$. It may appear then that the bounds on $m(n)$ are close together. But from a probabilistic standpoint a factor of 2^{n-1} may be considered as a unit. We could rewrite the problem as follows. Given a family F let X denote the number of monochromatic sets under a random coloring. What is the maximal $k=k(n)$ so that, if F is a family of n-sets with $E(X)\le k$, then $\Pr[X=0]>0$? In this formulation $cn^{1/3}<k(n)<cn^2$ and the problem clearly deserves more attention.