Algebraic techniques in combinatorial geometry

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17.06.2014

Preliminaries on Discrete (Combinatorial) Geometry

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- **What do we study?**

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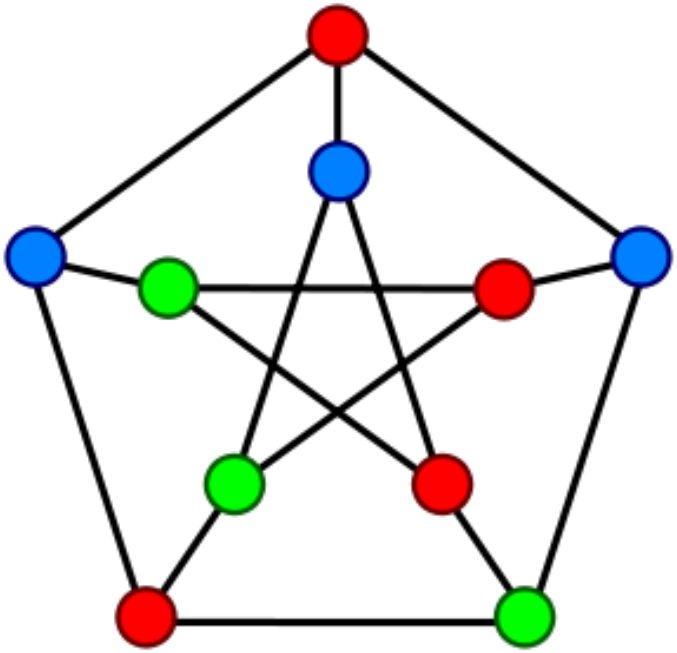
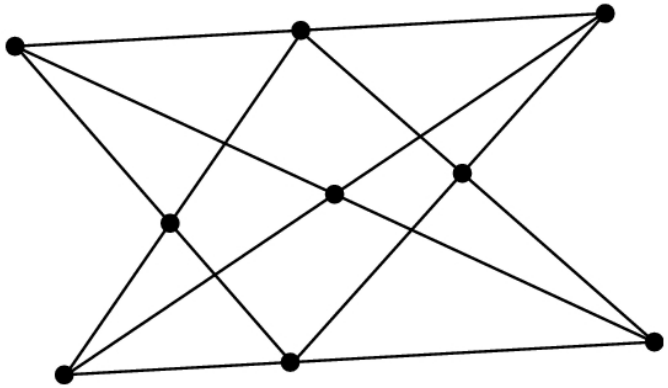
- Finite (discrete) sets of geometric objects (points, lines, polygons, polytopes, circles, planes) and their properties.





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What is the maximum number of times the unit distance can occur among a set of n points in the plane?

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Problem (Erdős)

What is the maximum number of times the unit distance can occur among a set of n points in the plane?

Theorem (Szemeredi-Trotter)

Given n points and m lines in the plane, the number of incidences between the set of points and the set of lines is of order

$$O(n^{2/3}m^{2/3} + m + n).$$

Why algebraic geometry?

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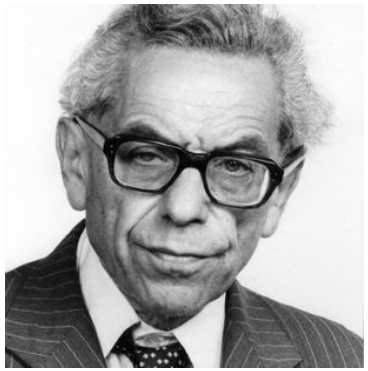
Prove that any set of n points in the plane determines at least $\Omega(n/\sqrt{\log n})$ distinct distances!



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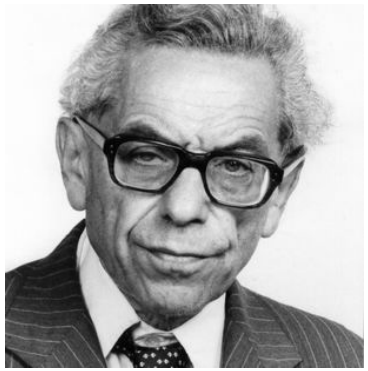

- **Solved...after more than 50 years! - 2010** - L.Guth, N.Katz - using tools from Algebraic geometry.

Tools from Algebraic Geometry...

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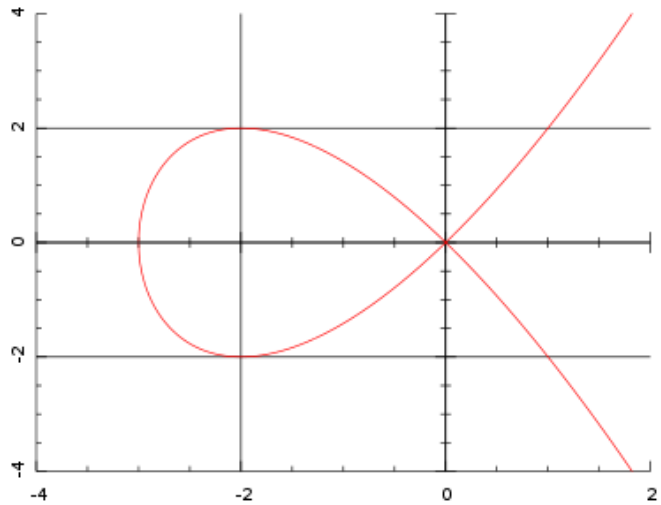
- **Definition (Algebraic curve).** We call C an **algebraic curve** if it is infinite and there exists a polynomial $f \in \mathbb{R}[x, y] \backslash \{0\}$ such that

$$C = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}.$$



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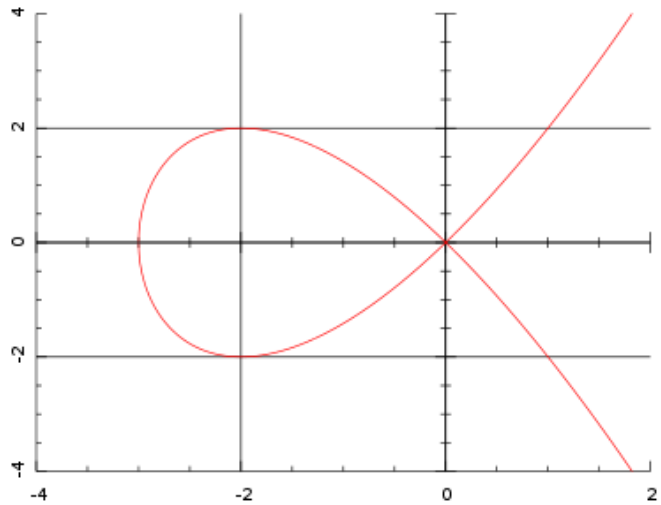

- **Lines and circles are algebraic curves!**

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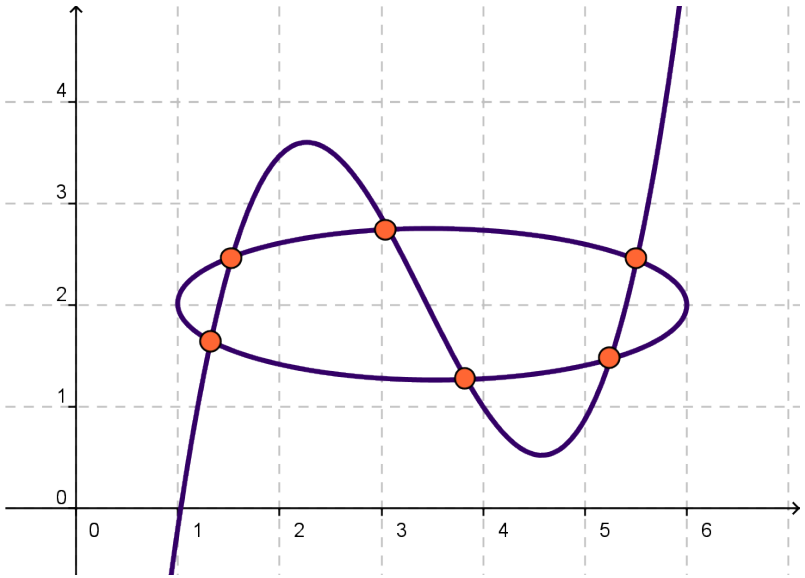
Theorem (Bézout's inequality)

Two algebraic curves of degrees d_1 and d_2 have at most $d_1 \cdot d_2$ intersection points, unless they have a common component.



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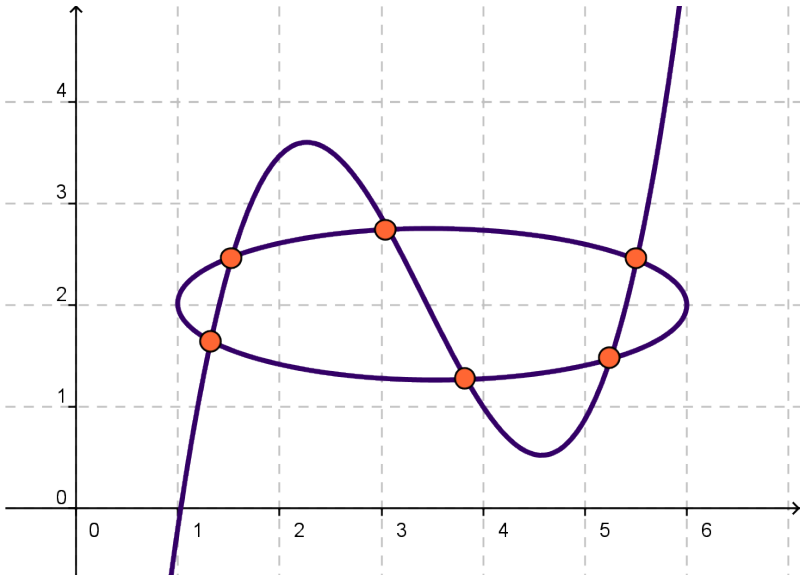

- The curve C is called **irreducible** if the polynomial f is irreducible over $\mathbb{R}$.

...and some applications in Discrete Geometry

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- **How to do this?**

...and some applications in Discrete Geometry

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- Given a polynomial f, we will denote by $Z(f)$ the zero set of f, namely

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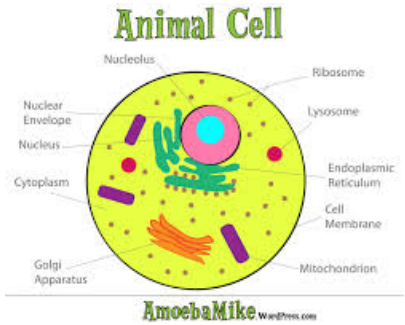
- Each connected component of $\mathbb{R}^2 \setminus Z(f)$ will be called **cell**.



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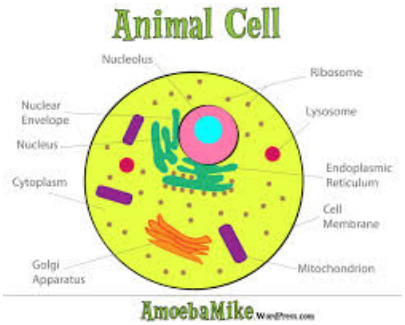

Definition (Partitioning polynomial)

Given a set P of n points in the plane, and $1 < r \leq n$ a parameter, we say that f is an r-**partitioning polynomial** for P if no connected component of $\mathbb{R}^2 \backslash Z(f)$ contains more than n/r points of P.

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Theorem

For every $r > 1$, every finite point set $P \subset \mathbb{R}^2$ admits an r-partitioning polynomial f of degree at most $O(\sqrt{r})$.

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- **The idea of polynomial partitioning method for bounding incidences between points and curves:** separately bound the number of incidences between the set of curves and the points from each connected component of $\mathbb{R}^2 \setminus Z(f)$, and respectively between the set of curves and the points on $Z(f)$.

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- Note that, for a given cell C_i, when counting incidences, we only have to consider the curves intersecting this cell.

Erdős' distinct distance problem

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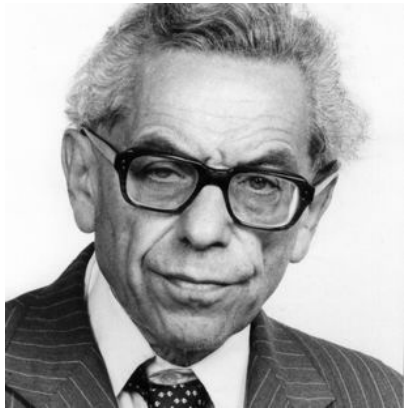
Problem (Erdős): Prove that any set of n points in the plane determines at least $\Omega(n/\sqrt{\log n})$ distinct distances.



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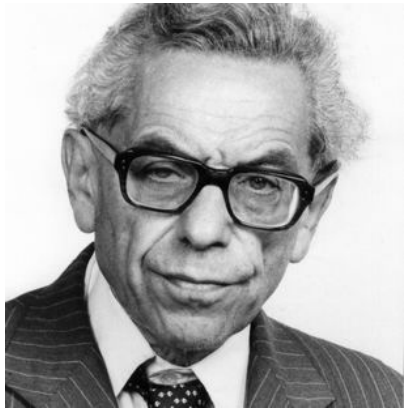

Let's see the idea behind the proof...

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- **Idea of proof** : Transform the problem to an incidence problem between points and curves (G.Elekes).

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- Use polynomial partitioning.

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Reduce bounding Q to bounding incidences between points and lines, which gives $|Q| \leq n^3 \log n$.

Variations of Erdős' distinct distance problem

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Theorem (Sharir-Sheffer-Solymosi, 2013)

Given two lines in $\mathbb{R}^2$, with n points each, the number of distinct distances between pairs of points (each point from a line) is

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Remark: If the two lines are orthogonal or parallel, there are constructions for which the number of distances is linear in n.

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Theorem (Pach-De Zeeuw, 2013)

Let C be a plane algebraic curve of degree d that does not contain a line or a circle. Then any set of n points on C determines at least $\Omega_d(n^{4/3})$ distinct distances.

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Remark: If the curve contains a line or a circle, then there are constructions for which the number of distances spanned is linear in n.

Distinct ”pinned” triangle areas

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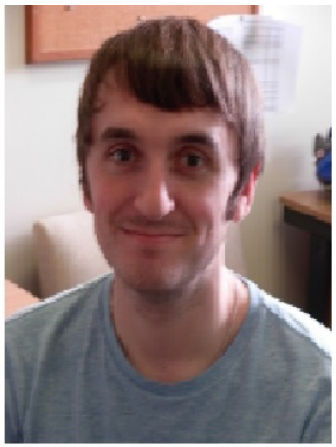
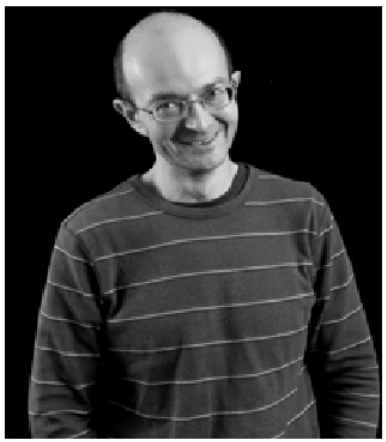
Theorem (Iosevich-Roche Newton-Rudnev, 2011)

A set of n points in $\mathbb{R}^2$ determines $\Omega(n/log(n))$ distinct pinned triangle areas, unless the points lie on a line passing through the origin.

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On algebraic curves...

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Theorem (Charalambides, 2013)

Given n points on an irreducible algebraic curve of degree d in $\mathbb{R}^2$, there are $\Omega_d(n^{5/4})$ distinct pinned triangle areas, unless the curve is a line, ellipse centred at the origin, or hyperbola centred at the origin.

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Theorem (Valculescu-De Zeeuw, 2014)

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A point $(q_s, q_t) \in P$ lies on C_{ij} iff $(p_i, p_j, q_s, q_t) \in Q$.

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Definition

Let $P \subset \mathbb{R}^D$ and let Γ be a set of curves in $\mathbb{R}^D$. We say that P and Γ form a system with k degrees of freedom and multiplicity M if any two curves in Γ intersect in at most M points of P, and any k points of P belong to at most M curves in Γ.

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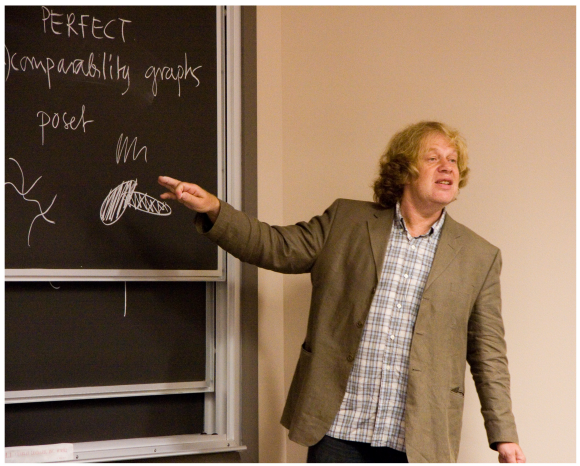
Theorem (Pach-Sharir)

If a set P of points in $\mathbb{R}^2$ and a set Γ of algebraic curves in $\mathbb{R}^2$ form a system with 2 degrees of freedom and multiplicity M, then

$$I(P,\Gamma) \leq C_M \cdot \max\{|P|^{2/3}|\Gamma|^{2/3}, |P|, |\Gamma|\},$$

where C_M is a constant depending only on M.



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However, we can show that after removing small subsets of points and curves they do form such a system.

What points and curves should we remove?

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Let

$$\Gamma_0 = \{C_{ij} \in \Gamma : \exists C_{kl} \in \Gamma : |C_{ij} \cap C_{kl}| = \infty\}.$$

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Lemma

If C is not exceptional, then for all $C_{ij}, C_{kl} \in \Gamma \backslash \Gamma_0$ we have

$$|C_{ij} \cap C_{kl}| \leq d^2,$$

and for all pairs of points in $P \backslash P_0$, there are at most d^2 curves in Γ that contain both points.

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Lemma

An irreducible algebraic curve of degree d has $\leq 4d$ linear automorphisms, unless it is a line or linearly equivalent to one of the following:
- an ellipse or a hyperbola centered at the origin;
- a parabola;
- a "pseudohyperbola" (algebraic curve of equation $x^p y^q = 1$, $p, q > 1$) or a "pseudocusp" (algebraic curve of equation $x^p y^{-q} = 1$, $p, q > 1$).

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Remark. Parabolas, pseudohyperbolas and pseudocusps can be excluded!

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Explicit constructions on the exceptional curves:

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Explicit constructions on the exceptional curves:

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The pinned triangle areas are of the form $\frac{1}{2}|2^{i-j} - 2^{j-i}|$ - **the number is linear in** n.

Open problems

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- Extend to distinct values of polynomials on algebraic curves.

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- Extend to higher dimensions.
- Extend to distinct values of polynomials on algebraic curves.
- Improve the exponent 4/3.

Thank you!

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