A course in algebraic combinatorial geometry

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Chapter 1

Introduction

We start with an overview of the two main problems that we will see in this course. Both have been partly solved (to different degrees), but each has also thrown up many new problems. In this course we will study those partial solutions, as well as some of the related problems.

Distinct distances. The following problem was posed by Erdős (remember that name) [7] in 1946:

How few distinct distances can a finite point set in $\mathbb{R}^2$ determine?

The distance referred to here is the Euclidean distance

$$D(p,q) = (x_p - x_q)^2 + (y_p - y_q)^2.$$

Of course, there is a square root missing there, but when considering the number of values of this function, the square root doesn't matter. For a point set $P \subset \mathbb{R}^2$, we write $D(P) = \{D(p,q) : p, q \in P\}$, and we write $D(n)$ for the minimum value of $D(P)$ over all n-point sets $P \subset \mathbb{R}^2$. The problem above thus asks to determine $D(n)$. That would be very difficult, and what Erdős had in mind was to find *asymptotic bounds* for $D(n)$. He proved right away that[1]

$$\sqrt{n} \lesssim D(n) \lesssim \frac{n}{\sqrt{\log n}}.$$

The lower bound comes from a quick proof that any set of points determines at least $\sqrt{n}$ distances, while the upper bound comes from a construction that has that many distances.

Over the years there were many improvements to the lower bound, up to $D(n) \gtrsim n^{0.86}$, proved by Katz and Tardos [24] in 2004 (building on a breakthrough of Solymosi and Tóth [40], who reached $n^{6/7}$ in 2001). These proofs were based on incidence geometry in the plane (more on that below) and topological graph theory (not a part of this course), and

[1] Throughout, we use the notation $f(n) \lesssim g(n)$ to mean that there is some constant C (independent of n) such that $f(n) \le C \cdot g(n)$ for all n that the functions are defined for. You can then guess what $\gtrsim$ means, and $\approx$ means that both $\lesssim$ and $\gtrsim$ hold. Other times we will use the notations O, Ω, Θ; $f(n) = O(g(n))$ is the same as $f(n) \lesssim g(n)$, $f(n) = \Omega(g(n))$ is the same as $f(n) \gtrsim g(n)$, and $f(n) = \Theta(g(n))$ is the same as $f(n) \approx g(n)$.

it seemed that these tools couldn't stretch much further. Then, in 2010, Guth and Katz jumped ahead all the way to

$$D(n) \gtrsim \frac{n}{\log n},$$

which essentially solved the problem, because it matches Erdős's upper bound up to a factor $\sqrt{\log n}$. The work of Guth and Katz [18] was particularly impactful because it introduced several new tools, mostly from algebraic geometry (with a touch of differential geometry). Actually, the algebraic geometry in these tools is fairly elementary; the real innovation was the way to use them in higher-dimensional incidence geometry. The first goal of this course will be to get to know these tools, and then to prove the Guth-Katz theorem; this will be the focus of the chapters up to Chapter 7.

Several variants of this problem are worth mentioning. First of all, one can of course ask the same question in higher dimensions. Write $D_d(n)$ for the minimum value of $D(P)$ over all n-point sets $P \subset \mathbb{R}^d$, where now D is the (squared) Euclidean distance function in $\mathbb{R}^d$. The best known estimates are

$$n^{\frac{2}{d}-\frac{1}{d^2}} \lesssim D_d(n) \lesssim n^{\frac{2}{d}}.$$

The upper bound comes from a construction, and the lower bound was proved by Solymosi and Vu [41]. In three dimensions, the next most interesting case, this can be improved to $n^{3/5}\log^{-1}(n) \lesssim D_3(n) \lesssim n^{2/3}$ using the Guth-Katz theorem. We will prove these bounds in Chapter 8.

In his 1946 paper [7], Erdős also introduced the *unit distance problem*: How often can a point set in $\mathbb{R}^2$ determine the *same* distance? Since it doesn't really matter what that distance is, the question is usually phrased in terms of unit distances. Let's write $U_d(P) := \{(p,q) \in P \times P : D(p,q) = 1\}$ for the number of unit distances determined by $P \subset \mathbb{R}^d$, and $U_d(n)$ for the maximum of $U_d(P)$ over all n-point sets $P \subset \mathbb{R}^d$. Erdős showed that

$$n^{1+c/\log\log n} \lesssim U_2(n) \lesssim n^{3/2}.$$

The upper bound was improved to $U_2(n) \lesssim n^{4/3}$ by Spencer, Szemerédi, and Trotter [44] in 1984, and it has been hopelessly stuck there ever since. This is now considered one of the biggest open problems in discrete geometry. We will see the proof of $U_2(n) \lesssim n^{4/3}$ in Chapter 5. In three dimensions, the best known bounds are $n^{4/3} \lesssim U_3(n) \lesssim n^{3/2}$, where the upper bound was obtained in [22, 51] using the new tools introduced by Guth and Katz (although [4] had come very close with other tools). We will prove these bounds in Chapter 8. Oddly enough, in dimensions larger than four this problem becomes trivial: $U_d(n) \approx n^2$ for $d \geq 4$. See Chapter 5 of Brass, Moser, and Pach [2] or Sheffer's more recent survey [36] for much more information about distance problems.

Sum-product problems. Our second type of problem is the *sumset-productset* problem (usually shortened to sum-product). It considers the *sumset* $A + A := \{a + a' : a, a' \in A\}$ and *productset* $A \cdot A := \{a \cdot a' : a, a' \in A\}$ of a set A in some ring. The main prototype is:

For $A \subset \mathbb{R}$, can both $|A + A|$ and $|A \cdot A|$ be small?

More precisely, write $SP_{\mathbb{R}}(n)$ for the minimum over all n-sets $A \subset \mathbb{R}$ of $\max\{|A+A|, |A \cdot A|\}$. The rationale is that if you take an *arithmetic progression*

$$A = \{a + ib : i = 1, \dots, n\},$$

then $|A + A| \lesssim n$, which is the smallest possible, but $|A \cdot A| \gtrsim n^{2-\varepsilon}$, which is almost the largest possible. On the other hand, for a *geometric progression*

$$B = \{a \cdot b^i : i = 1, \dots, n\},$$

we have $|B \cdot B| \lesssim n$ but $|B + B| \gtrsim n^{2-\varepsilon}$. These are the most extreme examples, but they suggest that it may be true that always either the sumset or the productset has size close to n^2.

This problem was first posed by Erdős and Szemerédi [13] in 1983, who proved right away that $SP_{\mathbb{R}}(n) \gtrsim n^{1+c}$ for some tiny constant c. Over the years, the c was slowly improved, until in 1997 Elekes [8] gave a shockingly simple proof of

$$SP_{\mathbb{R}}(n) \gtrsim n^{5/4}.$$

The proof made a surprising connection between this sum-product problem and incidence geometry, and it has inspired many later results. This wasn't the end of the story, because in 2009 Solymosi [38] proved that

$$SP_{\mathbb{R}}(n) \gtrsim \frac{n^{4/3}}{\log^{1/3} n},$$

using no incidence geometry, or in fact any other theory. We will see Elekes's proof as warmup in the next chapter.

As the notation suggests, one can ask this question over fields (or even rings) other than $\mathbb{R}$, like $\mathbb{C}$ or a finite field $\mathbb{F}_p$. Over $\mathbb{C}$, Solymosi's proof was extended by Konyagin and Rudnev [25] to obtain the same lower bound for $SP_{\mathbb{C}}(n)$. For a prime[2] field $\mathbb{F}_p$ (which some people consider to be the most interesting), there have been many different bounds. These are a bit complicated to state, because they usually only hold for $|A|$ in certain ranges relative to p. We just mention a recent result of Roche-Newton, Rudnev, and Shkredov [30], using an incidence bound of Rudnev [29], that in turn used the theorem of Guth and Katz (which is why it's relevant to this course). They proved that $SP_{\mathbb{F}_p}(A) \gtrsim |A|^{6/5}$ for $|A| < p^{5/8}$.

An algebraically interesting variant of this problem is to ask the same question for more general polynomials than $x + y$ or $x \cdot y$. For a polynomial $F \in \mathbb{R}[x, y]$, we write $F(A, A) := \{F(a, a') : a, a' \in A\}$. Then one can for instance ask for a constant c so that

$$\max\{|A + A|, |F(A, A)|\} \gtrsim |A|^{1+c}.$$

What makes this problem difficult is that there are clearly some polynomials for which no such c exists; consider for instance $F(x, y) = x + y$. Shen [37] proved that the only

[2]We will use p to denote an odd prime without further comment (except when it denotes a point, but this should be clear from the context). Much of what we say may work for general finite fields, but for simplicity we restrict to $\mathbb{F}_p$ in these notes.

exceptional polynomials are of the form $F(x,y) = G(ax+by)$, and otherwise one can take $c = 1/4$. We will prove this in Chapter ???.

Actually, it turns out that for most polynomials there is such a lower bound regardless of $|A+A|$. It was proved by Elekes and Rónyai [11] in 2000 that for $F \in \mathbb{R}[x,y]$ one has

$$|F(A,A)| \gtrsim |A|^{1+\varepsilon},$$

unless F has the special form $F(x,y) = F(H(x)+K(y))$ or $F(x,y) = G(H(x)\cdot K(y))$. The proof again used incidence geometry and algebraic methods. The lower bound of Elekes and Rónyai was recently improved to $|A|^{4/3}$ by Raz, Sharir, and Solymosi [32], with a proof partly inspired by that of Guth and Katz. We will see it in Chapter ???.

Incidence geometry. Although it is not one of the "two problems" in the title of this section, we have mentioned incidence geometry several times, so we should say at least something about what it is. The prototypical incidence theorem was proved by Szemerédi and Trotter [45] in 1983. It states that given a set $\mathcal{P} \subset \mathbb{R}^2$ and a set $\mathcal{L}$ of lines, the set of *incidences*

$$I(\mathcal{P},\mathcal{L}) := \{(p,\ell) \in \mathcal{P}\times\mathcal{L} : p \in \ell\}$$

satisfies

$$|I(\mathcal{P},\mathcal{L})| \lesssim |\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + |\mathcal{P}| + |\mathcal{L}|.$$

This is a central theorem in combinatorial geometry. Many similar bounds have since been proved for incidences between all kinds of geometric objects. Most of the theorems mentioned above involve incidence bounds in some way, and we will spend much of this course on them.

Chapter 2

Warmup theorems

As a warmup for efforts to come, this chapter shows some simplified versions of the theorems that we will see later on. The proofs in the first three sections are entirely self-contained, in the sense that we don't use any serious tools from outside this chapter. On the other hand, they introduce a number of ideas that will show up repeatedly in this course.

2.1 Szemerédi-Trotter for beginners

Lemma 2.1. *Let $A, B \subset \mathbb{R}$ with $|A| = |B|$, set $\mathcal{P} = A \times B$, and let $\mathcal{L}$ be a set of lines in $\mathbb{R}^2$. Then*

$$|I(\mathcal{P}, \mathcal{L})| \lesssim |\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + |\mathcal{P}| + |\mathcal{L}|.$$

Proof. Let r be some integer that we choose at the end of the proof. Partition $\mathbb{R}$ by cutting at r points $\alpha_1, \ldots, \alpha_r$ not in A, so that each of the resulting $r+1$ intervals contains at most $2|A|/r$ points of A.[1] This is possible as long as r satisfies $1 \leq r \leq |A| = |\mathcal{P}|^{1/2}$. Similarly, cut $\mathbb{R}$ at $\beta_1, \ldots, \beta_r$, so that each of the resulting $r+1$ intervals contains at most $2|B|/r$ points of B. Then the $2r$ lines $x = \alpha_i$ and $y = \beta_j$ are disjoint from $\mathcal{P}$, and they partition $\mathbb{R}^2$ into $(r+1)^2$ cells (open rectangles), each containing at most $(2|A|/r) \cdot (2|B|/r) = 4|\mathcal{P}|/r^2$ points of $\mathcal{P}$. Since two lines intersect in at most one point, any line in $\mathcal{L}$ intersects at most $2r+1$ of the cells, since to pass from one cell to another, it has to intersect one of the $2r$ partitioning lines.

Let $\mathcal{I}_1$ be the set of incidences (p, ℓ) such that ℓ has no other incidences in the cell that contains p, and let $\mathcal{I}_2$ be the set of incidences (p, ℓ) such that ℓ has at least two incidences in the cell that contains p. Then

$$|\mathcal{I}_1| \leq (2r+1)|\mathcal{L}| \lesssim r|\mathcal{L}|.$$

Given two points in a cell, there is at most one line of $\mathcal{L}$ that contains both. Since a cell has at most $4|\mathcal{P}|/r^2$ points, it contains at most $\binom{4|\mathcal{P}|/r^2}{2} \leq 8|\mathcal{P}|^2/r^4$ pairs of points. Each pair of points gives at most two incidences, so inside a cell there can be at most $16|\mathcal{P}|^2/r^4$ incidences from $\mathcal{I}_2$. Thus, summing over all cells,

$$|\mathcal{I}_2| \leq (r+1)^2 \cdot \frac{|\mathcal{P}|^2}{r^4} \lesssim \frac{|\mathcal{P}|^2}{r^2}.$$

[1] The factor 2 is just there so that we don't have to worry about what happens if $|A|/r$ is not an integer.

Altogether we have
$$|I(\mathcal{P},\mathcal{L})| \lesssim r|\mathcal{L}| + \frac{|\mathcal{P}|}{r^2}.$$
Now we choose r to minimize this sum. Equating $r|\mathcal{L}| = |\mathcal{P}|^2/r^2$ gives $r = |\mathcal{P}|^{2/3}/|\mathcal{L}|^{1/3}$ (actually, we should choose r to be an integer close to this value). Plugging this in, we get
$$|I(\mathcal{P},\mathcal{L})| \lesssim |\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3}.$$

This only works when $1 \le r \le |\mathcal{P}|^{1/2}$; the other cases we treat separately. First suppose that $r < 1$, which implies $|\mathcal{L}| > |\mathcal{P}|^2$. The lines that have at most one point on them give altogether at most $|\mathcal{L}|$ incidences. On the other hand, there are less than $|\mathcal{P}|^2/2$ pairs of distinct points, and each pair is contained in at most one line of $\mathcal{L}$, so the lines that contain at least two points give altogether at most $|\mathcal{P}|^2 < |\mathcal{L}|$ incidences. So in this case, all the incidences are covered by the term $|\mathcal{L}|$. A similar argument applies for the case $r > |\mathcal{P}|^{1/2}$, or $|\mathcal{L}| < |\mathcal{P}|^{1/2}$: The points contained in at most one line give at most $|\mathcal{P}|$ incidences, while the number of incidences contained in at least two lines is at most $|\mathcal{L}|^2 < |\mathcal{P}|$. So adding the terms $|\mathcal{P}| + |\mathcal{L}|$ covers all the cases. $\square$

2.2 Elekes's sum-product theorem

Theorem 2.2. *For any $A \subset \mathbb{R}$ we have*
$$\max\{|A+A|, |A\cdot A|\} \gtrsim |A|^{5/4}.$$

Proof. Assume for now that $|A+A| = |A\cdot A|$. Define the point set
$$\mathcal{P} := (A+A) \times (A\cdot A) \subset \mathbb{R}^2.$$
Given $a, b \in A$, define a line ℓ_{ab} by $y = b(x-a)$, and define
$$\mathcal{L} := \{\ell_{ab} : a, b \in A\}.$$
Each line of $\mathcal{L}$ hits at least $|A|$ points of $\mathcal{P}$, because $(a'+a, ba') \in \ell_{ab}$ for each $a' \in A$. So there are at least $|\mathcal{L}|\cdot|A| = |A|^3$ incidences.

Applying Lemma 2.1 gives
$$|A|^3 \le I(\mathcal{P},\mathcal{L}) \lesssim (|A+A||A\cdot A|)^{2/3}(|A|^2)^{2/3} + |A+A||A\cdot A| + |A|^2.$$
The third term on the right cannot be bigger than $|A|^3$, the second would give an even stronger bound, and if the first dominates, then we get
$$|A+A||A\cdot A| \gtrsim |A|^{5/2}.$$

The assumption that the two factors $|A+A|$ and $|A\cdot A|$ of $\mathcal{P}$ have the same size need not hold. However, we can just add random numbers to the smaller one until they are equal, without affecting the proof above. The outcome is that both have size $\gtrsim |A|^{5/4}$, so the larger, unmodified, one has that size. $\square$

2.3 Distances between lines

Lemma 2.1 does not strongly use the fact that the lines are lines; it only uses that two lines intersect in a bounded number of points, that two points are contained in a bounded number of lines, and that a line intersects a partitioning line in a bounded number of points. Therefore it is not hard to see that the same bound follows for any class of algebraic curves that satisfy the same properties. For instance, we can obtain the following lemma (we will prove something much stronger later on).

Lemma 2.3. *Let $A, B \subset \mathbb{R}$ with $|A| = |B|$, set $\mathcal{P} = A \times B$, and let $\mathcal{C}$ be a set of hyperbolas in $\mathbb{R}^2$, such that any two hyperbolas from $\mathcal{C}$ intersect in at most two points of $\mathcal{P}$, and any two points of $\mathcal{P}$ are contained in at most two hyperbolas from $\mathcal{C}$. Then*

$$|I(\mathcal{P}, \mathcal{C})| \lesssim |\mathcal{P}|^{2/3}|\mathcal{C}|^{2/3} + |\mathcal{P}| + |\mathcal{C}|.$$

With this incidence bound we can prove the following theorem, where similarly to before we write $D(P_1, P_2) := \{D(p_1, p_2) : p_1 \in P_1, p_2 \in P_2\}$.

Theorem 2.4. *Let L_1, L_2 be two lines in $\mathbb{R}^2$, and $P_1 \subset L_1$ and $P_2 \subset L_2$ with $|P_1| = |P_2| = n$. Then*

$$|D(P_1, P_2)| \gtrsim n^{4/3},$$

unless L_1 and L_2 are parallel or orthogonal.

Proof. Assume the lines are not parallel or orthogonal. Translate their intersection point to the origin, and rotate so that one of the lines is the x-axis; this will not affect the number of distances. Then the lines are $L_1 : y = 0$ and $L_2 : y = mx$ for some $m \neq 0$.

The strategy of the proof is to count the following set of quadruples:

$$Q := \{(p, p', q, q') \in P_1^2 \times P_2^2 : D(p, q) = D(p', q')\}.$$

We can get a lower bound for Q in terms of $|D(P_1, P_2)|$, as follows. We first note that any quadruple in Q consists of two pairs from $D^{-1}(a) = \{(p, q) \in L_1 \times L_2 : D(p, q) = a\}$ for the same a; then we apply the Cauchy-Schwarz inequality; and finally we use that the sum of $|D^{-1}(a)|$ over all $a \in D(P_1, P_2)$ equals n^2. Thus

$$|Q| = \sum_{a \in D(P_1, P_2)} |D^{-1}(a)|^2 \geq \frac{1}{|D(P_1, P_2)|} \left(\sum_{a \in D(P_1, P_2)} |D^{-1}(a)| \right)^2 = \frac{n^4}{|D(P_1, P_2)|}.$$

On the other hand, we will get an upper bound $|Q| \lesssim n^{8/3}$, which combined with the inequality above will give $|D(P_1, P_2)| \gtrsim n^4/|Q| \gtrsim n^{4/3}$.

For each $(q, q') \in P_2^2$, we define a curve

$$C_{qq'} := \{(p, p') \in L_1 \times L_2 : D(p, q) = D(p', q')\},$$

which lies in the plane $L_1 \times L_1$. Then we have $(p, p') \in C_{qq'}$ if and only if $(p, p', q, q') \in Q$. If we set $\mathcal{P} = P_1 \times P_1$ and $\mathcal{C} = \{C_{qq'} : (q, q') \in P_2^2\}$, then $|Q| = |I(\mathcal{P}, \mathcal{C})|$.

We can parametrize the plane $L_1 \times L_1$ by $p = (x, 0), p' = (y, 0)$, so that it becomes the xy-plane. If we set $q = (s, ms), q' = (t, mt)$, the equation $D(p, q) = D(p', q')$ becomes

$$(x - s)^2 + m^2 s^2 = (y - t)^2 + m^2 t^2.$$

For $s \neq \pm t$, this equation describes a hyperbola in the xy-plane. When $s = \pm t$, it is not a hyperbola but a pair of lines, and we will have to treat this case separately. Note that different (s, t) give distinct hyperbolas, for instance because the coefficient of x^2 is 1, but that of x is $-2s$ and that of y is $-2t$.

Set $\mathcal{P} = P_1 \times P_1$ and $\mathcal{C}_1 = \{C_{qq'} : (q, q') \in P_2 \times P_2, q \neq \pm q'\}$. The curves in $\mathcal{C} \backslash \mathcal{C}_1$ are the exceptions which are not hyperbolas. The number of incidences coming from them is at most $4n^2$, because for each $q \in L_2$, we have $|C_{qq} \cap (P_1 \times P_1)| \leq 2n$, and similarly for $q' = -q$. If we can apply Lemma 2.3 to $\mathcal{C}_1$, we would get

$$I(\mathcal{P}, \mathcal{C}_1) \lesssim (n^2)^{2/3}(n^2)^{2/3} + n^2 + n^2 \lesssim n^{8/3}.$$

Thus $I(\mathcal{P}, \mathcal{C}) = I(\mathcal{P}, \mathcal{C}_1) + I(\mathcal{P}, \mathcal{C}_2) \lesssim n^{8/3} + n^2 \lesssim n^{8/3}$, which would complete the proof.

To prove that the curves in $\mathcal{C}_1$ satisfy the two conditions of Lemma 2.3, observe that a curve passing through (x_1, y_1) and (x_2, y_2) corresponds to a solution (s, t) of a system

$$(x_1 - s)^2 + m^2 s^2 = (y_2 - t)^2 + m^2 t^2,$$
$$(x_2 - s)^2 + m^2 s^2 = (y_2 - t)^2 + m^2 t^2.$$

Subtracting these two equations gives the equation of a line, and a line and a hyperbola intersect in at most two points, so this system has at most two solutions. A similar argument gives that any two of these hyperbolas intersect in at most two points. □

The exponent in the lower bound is most likely not tight, and improving it is an interesting open problem. On the other hand, the exceptions are what they should be, because in those cases there are sets for which $|D(P_1, P_2)| \approx n$. If the lines are parallel, say $y = 0$ and $y = 1$, take the point sets $P_1 := \{(i, 0) : i = 1, \ldots, n\}$ and $P_2 := \{(j, 1) : j = 1, \ldots, n\}$. If the lines are orthogonal, say $y = 0$ and $x = 0$, take the point sets $P_1 := \{(\sqrt{i}, 0) : i = 1, \ldots, n\}$ and $P_2 := \{(0, \sqrt{j}) : j = 1, \ldots, n\}$.

Notes. Lemma 2.1 is a simplified version of the Szemerédi-Trotter Theorem, which we prove in full in Chapter 5. The earliest reference where I have seen this idea – giving a quick proof of an incidence theorem when the point set is a Cartesian product – is Solymosi and Vu [42]. We will use the same idea in Chapter 9 to prove a theorem of Solymosi and De Zeeuw [43], to prove a theorem about incidences on a complex Cartesian product.

Theorem 2.2 was proved by Elekes in [8], using the full Szemerédi-Trotter Theorem. The history of this problem was already described in the Introduction.

Theorem 2.4 was proved in Sharir, Sheffer, and Solymosi [34]. The problem was introduced by Purdy (see [2, Chapter 5.5]). The first superlinear bound followed from the theorem of Elekes and Rónyai [11] (see Chapter ???), and Elekes [9] then specialized that proof to get the lower bound $n^{5/4}$. He conjectured [9] that this bound can still be improved a lot.

Problem 2.5 (Elekes). *For n-point sets $P_1 \subset L_1, P_2 \subset L_2$ we have $|D(\mathcal{P}_1, \mathcal{P}_2)| \gtrsim n^{2-\varepsilon}$, unless L_1, L_2 are parallel or orthogonal.*

Interlude: The Elekes-Sharir framework

We now give a sketch of the Elekes-Sharir framework, which connects distances in $\mathbb{R}^2$ to incidences of lines in $\mathbb{R}^3$. It was introduced by Elekes and Sharir [12], and it was used by Guth and Katz to tackle the distinct distances problem in $\mathbb{R}^2$. We sketch it here as motivation for the next few chapters, where we will obsess about lines in $\mathbb{R}^3$; later we will do it more carefully and in a more general way.

The goal is to prove that $|D(P)| \gtrsim |P|/\log|P|$ for any $P \subset \mathbb{R}^2$. As in the proof of Theorem 2.4, we will do this by finding lower and upper bounds for the size of the set of quadruples

$$Q := \{(p, q, r, s) \in P^4 : D(p, q) = D(r, s)\}.$$

Then we have

$$|Q| = \sum_{a \in D(P)} |D^{-1}(a)|^2 \geq \frac{1}{|D(P)|} \left(\sum_{a \in D(P)} |D^{-1}(a)| \right)^2 = \frac{|P|^4}{|D(P)|}.$$

So we want to prove that $|Q| \lesssim |P|^3 \log|P|$, since then $|D(P)| \gtrsim |P|/\log|P|$.

Note that at this point the approach from Theorem 2.4 would not work so well. We could define a variety $V_{pr} := \{(q, s) \in \mathbb{R}^2 \times \mathbb{R}^2 : D(p, q) = D(r, s)\}$ and try to bound the incidences with the points $\{(q, s) \in P \times P\}$. However, this would be an incidence problem for three-dimensional hypersurfaces in $\mathbb{R}^4$, which is much more difficult. Some bounds for such problems are known, but even if they could be applied here, they give a weaker result.

Instead, Elekes suggested to consider the *partial symmetries* of P, i.e., the isometries (distance-preserving transformations) that map some points of P to other points of P. Let $\mathcal{R}$ be the set of *rotations* of $\mathbb{R}^2$. Observe that, more or less,[2]

$$(p, q, r, s) \in Q \iff \exists R \in \mathcal{R} \text{ such that } R(p) = r, R(q) = s.$$

Therefore, to count quadruples, one can try to count rotations instead.

Somewhat like in Theorem 2.4, we fix two points in a quadruple, and consider the sets

$$L_{pr} := \{R \in \mathcal{R} : R(p) = r\},$$

for all $(p, r) \in P \times P$. Then we have (again with a grain of salt)

$$(p, q, r, s) \in Q \iff L_{pr} \cap L_{qs} \neq \emptyset.$$

[2] In this sketch, we'll ignore the other isometries, i.e. translations and reflections. In the stated equivalence, one should also include translations, but they can be easily discounted in the subsequent argument.

Note that the set of rotations is three-dimensional (each rotation corresponds to a fixed point and an angle), and L_{pr} should be a one-dimensional subset (because it is defined by two equations, one for the x-coordinate and one for the y-coordinate). So we have arrived at an incidence problem for curves in some three-dimensional space. Roughly, the number of quadruples equals the number of intersecting pairs of lines, and we want to show that this number is bounded above by $|P|^3 \log |P|$.

To make this more concrete, we choose coordinates for the set of rotations. Given a rotation R, write (a, b) for its fixed point and θ for its angle. Then we can write (abusing notation a little)

$$\mathcal{R} = \{(a, b, \cot(\theta/2)) \in \mathbb{R}^3\}.$$

The great thing about this parametrization (in particular the choice of the cotangent) is that the sets L_{pr} become *lines*.[3] To see this, first assume that $p = (0, 1)$ and $r = (0, -1)$. Then a rotation that sends p to r must have its fixed point on the x-axis, so it equals $(t, 0)$ for some t. If θ is the rotation angle, then the definition of the cotangent gives $\cot(\theta/2) = t$. Thus we can write

$$L_{pr} = \{(t, 0, t) : t \in \mathbb{R}\},$$

which is indeed a line. For a general L_{pr}, we can scale the plane so that $D(p, r) = 2$, and then move p and r to $(0, 1)$ and $(0, -1)$. It is not hard to see that this will not affect whether or not L_{pr} is a line.

Thus we want to bound the number of intersecting pairs of lines from the set

$$\mathcal{L} := \{L_{pr} : (p, r) \in P \times P\}.$$

It turns out to be more convenient to count intersection points, distinguishing the points by how many lines intersect at them. So we define

$$\mathcal{I}_{\geq k}(\mathcal{L}) := \{p \in \mathbb{R}^3 : p \text{ lies in at least } k \text{ lines of } \mathcal{L}\}.$$

Guth and Katz proved the following two theorems; the first is essentially the case $k = 2$ of the second, except that it requires an extra condition (which we will explain later), and has a very different proof.

Theorem (Guth-Katz A). *Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$ such that no $|\mathcal{L}|^{1/2}$ are in a plane or quadric surface (an algebraic surface of degree two). Then*

$$|\mathcal{I}_{\geq 2}(\mathcal{L})| \lesssim |\mathcal{L}|^{3/2}.$$

Theorem (Guth-Katz B). *Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$ such that no $|\mathcal{L}|^{1/2}$ are in a plane. Then for $3 \leq k \leq |\mathcal{L}|^{1/2}$ we have*

$$|\mathcal{I}_{\geq k}(\mathcal{L})| \lesssim \frac{|\mathcal{L}|^{3/2}}{k^2}.$$

[3] Elekes and Sharir [12] used a different parametrization and obtained curves that were less pleasant than lines. It was a small but crucial step for Guth and Katz to choose the parametrization so that these curves become lines.

Note that we could see Theorem A as the case $k = 2$ of Theorem B, so that the bound $|\mathcal{I}_{\geq k}(\mathcal{L})| \lesssim |\mathcal{L}|^{3/2}/k^2$ holds for $2 \leq k \leq |\mathcal{L}|^{1/2}$. Nevertheless, we distinguish the parts because they have very different proofs.

We will have to show that the lines in $\mathcal{L}$ satisfy the conditions of these two theorems, but we omit that in this sketch. Combining the theorems, we finally get

$$\begin{aligned}
|\{(L, L') \in \mathcal{L} \times \mathcal{L} : L \cap L' \neq \emptyset\}| &= \sum_{k=2}^{|\mathcal{L}|^{1/2}} (|\mathcal{I}_{\geq k}(\mathcal{L})| - |\mathcal{I}_{\geq k+1}(\mathcal{L})|) \cdot k^2 \\
&\lesssim \sum_{k=2}^{|\mathcal{L}|^{1/2}} |\mathcal{I}_{\geq k}(\mathcal{L})| \cdot k \lesssim \sum_{k=2}^{|\mathcal{L}|^{1/2}} \frac{|\mathcal{L}|^{3/2}}{k^2} \cdot k \\
&\lesssim |\mathcal{L}|^{3/2} \cdot \sum_{k=2}^{|\mathcal{L}|^{1/2}} \frac{1}{k} \lesssim |\mathcal{L}|^{3/2} \log |\mathcal{L}|.
\end{aligned}$$

This gives

$$|D(P)| \gtrsim \frac{|P|^4}{|Q|} \gtrsim \frac{|P|^4}{|\mathcal{L}|^{3/2} \log |\mathcal{L}|} \gtrsim \frac{|P|^4}{(|P|^2)^{3/2} \log |P|^2} \gtrsim \frac{|P|}{\log |P|},$$

completing the proof sketch.

Chapter 3

Triple intersection points of lines in $\mathbb{R}^3$

The purpose of this chapter may seem a bit confusing. We prove a special case of the Guth-Katz Theorem B, but as an introduction to Theorem A, while all cases of Theorem B will later be covered by a different proof. The reason is that Theorem A, the case $k = 2$, is the hardest part of the proof of Guth and Katz, and the proof of the case $k = 3$ is a great introduction to it. Before that, we prove a subcase of the case $k = 3$, where we only bound the number of joints, which are triple intersection points where the lines do not lie in a common plane. This subcase is a nice model for the proofs that follow.

Although the purpose is to obtain these results in $\mathbb{R}^3$, the proofs also work in $\mathbb{C}^3$ (unlike the proof of Theorem B), and in some ways actually become easier. The statements in $\mathbb{R}^3$ then follow directly.

The following construction shows that the bounds in this chapter are tight.

Construction 3.1. Let $\mathcal{L}_1$ be the set of all lines of the form $\{(i, j, t) : t \in \mathbb{C}\}$ for $i, j \in [n]$, let $\mathcal{L}_2$ be the set of all lines $\{(i, t, k) : t \in \mathbb{C}\}$ for $i, k \in [n]$, and let $\mathcal{L}_3$ be the set of all lines $\{(t, j, k) : t \in \mathbb{C}\}$ for $j, k \in [n]$. Set $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. Then $\mathcal{L}$ has $3n^2$ lines, and it has $n^3 = \Theta(|\mathcal{L}|)^{3/2}$ triple intersection points, all of which are joints.

3.1 Joints

Given a set of lines in $\mathbb{F}^3$ for some field $\mathbb{F}$, a point is called a **joint** if it is an intersection point of at least three lines, *and* these lines do not all lie in the same plane. In what follows we will focus on $\mathbb{C}^3$, but most of it works over any field and in any dimension (greater than two). We will show that a set $\mathcal{L}$ of lines in $\mathbb{C}^3$ determines $O(|\mathcal{L}|^{3/2})$ joints. Soon after we will show the same bound for arbitrary intersection points (with certain exceptions), but the proof for joints is simpler, and it provides a good model for the more difficult proofs to come. We first give a rough sketch of the proof, and then we will make it more precise.

Proof sketch. Let $\mathcal{L}$ be a set of lines in $\mathbb{C}^3$, write $|\mathcal{L}| = N$, and let J be the set of joints determined by $\mathcal{L}$. Assume for contradiction that $|J| = CN^{3/2}$ for some large C. By interpolation, there is a polynomial $F \in \mathbb{C}[x, y, z]$ such that $\deg(F) \lesssim |J|^{1/3} = C^{1/3}N^{1/2}$ and $J \subset Z(F)$. On average, a line of $\mathcal{L}$ contains $|J|/|\mathcal{L}| = CN^{1/2}$ joints. Let's assume, for the sake of this sketch, that all lines are average, so a line $\ell \in \mathcal{L}$ intersects the surface

$Z(F)$ in at least $CN^{1/2}$ points. Because the degree of F is only $C^{1/3}N^{1/2}$, this must mean that $\ell \subset Z(F)$. This holds for every line in $\mathcal{L}$, so in fact $\cup\mathcal{L} \subset Z(F)$.

Now consider a joint $p \in Z(F)$. There are three non-coplanar lines that intersect at p and are contained in $Z(F)$. Then p must be a singularity of $Z(F)$, since there cannot be a tangent plane at p. This means that the gradient $\nabla F = (F_x, F_y, F_z)$ vanishes at p. We have this for all joints, so it follows that $J \subset Z(F_x)$. Since $\deg(F_x) < \deg(F)$, we have found a lower-degree polynomial that interpolates J. But we can assume from the start that F is minimal, and then we have a contradiction. So we must have $|J| < CN^{3/2}$. $\square$

We now state the tools that we need to make this argument precise; proofs can be found in the appendix. The pruning lemma is proved in Section E.2, the interpolation lemma is explained in Section B.1, and the third lemma, which is a corollary of Bézout's inequality, is proved in Section A.2.

Lemma 3.2 (Pruning). *Let G be a bipartite graph with vertex sets A and B, such that B has no isolated vertices. Then there are nonempty $A' \subset A, B' \subset B$ such that in the induced subgraph G' on $A' \cup B'$, every $a \in A'$ has $\deg_{G'}(a) \geq |B|/(2|A|)$, and every $b \in B'$ has $\deg_{G'}(b) = \deg_G(b)$.*

Lemma 3.3 (Interpolation). *Let $S \subset \mathbb{C}^3$ be a finite set. Then there is $F \in \mathbb{C}[x_1, \dots, x_D]\backslash\mathbb{C}$ such that $S \subset Z_{\mathbb{C}^D}(F)$ and $\deg(F) \leq 3|S|^{1/3}$.*

Lemma 3.4 (Bézout). *If L is a line in $\mathbb{C}^3$ and $Z(F) \subset \mathbb{C}^3$, then $|Z(F) \cap L| \leq \deg(F)$ or $L \subset Z(F)$.*

We can now prove the theorem in full detail.

Theorem 3.5. *A set $\mathcal{L}$ of lines in $\mathbb{C}^3$ determines $O(|\mathcal{L}|^{3/2})$ joints.*

Proof. Write $|\mathcal{L}| = N$ and let J be the set of joints determined by $\mathcal{L}$. Assume $|J| = 2CN^{3/2}$. By pruning (Lemma 3.2 or E.3), there are subsets $\mathcal{L}' \subset \mathcal{L}$ and $J' \subset J$ such that every line $\ell \in \mathcal{L}'$ contains at least $2CN^{3/2}/(2N) = CN^{1/2}$ points of J', and for every point $p \in J'$, each line of $\mathcal{L}$ that contains p is in $\mathcal{L}'$, so p also forms a joint of $\mathcal{L}'$.

By interpolation (Lemma 3.3 or B.1), there exists $F \in \mathbb{C}[x, y, z]$ such that

$$J' \subset Z(F)$$

and $\deg(F) \leq 3|J'|^{1/3} \leq 6C^{1/3}N^{1/2}$. We can assume that F is a minimum-degree polynomial with this property.

For any line ℓ, Lemma 3.4 (or Lemma A.7) tells us that either $|\ell \cap Z(F)| \leq \deg(F) \leq 6C^{1/3}N^{1/2}$ or $\ell \subset Z(F)$. For $\ell \in \mathcal{L}'$ we have $|\ell \cap Z(F)| \geq |\ell \cap J'| \geq CN^{1/2}$, so we must have $\ell \subset Z(F)$ (assuming $C > 6C^{1/3}$). This holds for all $\ell \in \mathcal{L}'$, so

$$\cup\mathcal{L}' \subset Z(F).$$

Consider a joint $p \in J'$. It is contained in $Z(F)$, and there must also be three non-coplanar lines through p contained in $Z(F)$. We claim that this implies $(\nabla F)(p) = 0$.

Indeed, parameterize one of these lines as $p + tv$ for a unit vector $v = (v_1, v_2, v_3)$. Then $F(p + tv)$, as a polynomial in t, is identically zero, so in particular

$$0 = \frac{\partial}{\partial t} F(a + tv) = F_x \cdot v_1 + F_y \cdot v_2 + F_z \cdot v_3 = \nabla F \cdot v.$$

Doing this for all three lines shows that $(\nabla F)(p)$ is orthogonal to three independent vectors, which implies it is the zero vector.

So $J' \subset Z(F_x)$ (say). But $\deg(F_x) < \deg(F)$, contradicting the minimality of F. $\square$

In the proof above we could also have interpolated the lines instead of the joints, using the following lemma, which is proved in Section B.1.

Lemma 3.6. *Let $\mathcal{L}$ be a finite set of lines in $\mathbb{C}^3$. Then there exists $F \in \mathbb{C}[x, y, z] \backslash \mathbb{C}$ such that $\cup \mathcal{L} \subset Z(F)$ and $\deg(F) \leq 9|\mathcal{L}|^{1/2}$.*

This would have made the start of the proof somewhat easier as we could have skipped the pruning step as well as the application of Bézout. On the other hand, we would assume F to be minimal with the property that it interpolates the lines, and we would want to get a contradiction. Since we get that F_x vanishes on all the joints, we would still have to show F_x vanishes on all the lines, and for that we would need pruning and Bézout anyway. Nevertheless, it's worth noting this alternative approach.

3.2 Triple intersection points

Let's try to apply the approach we used for joints to arbitrary triple intersection points. We used the gradient as a polynomial (or vector of polynomials) that detects joints on a surface that contains the set of lines. We'll see below that there is also a polynomial that vanishes at any intersection point of three lines that are contained in a surface. However, the big difference is that, unlike the gradient, this polynomial has *larger* degree than the surface, which ruins the proof-by-minimality that worked so well for joints. Because of this we will have to do something slightly more complicated.

In fact, it makes sense that the same approach does not work, because for triple points we expect an exception when there are many lines in a plane. The following construction shows why.

Construction 3.7. Let $\mathcal{L}$ consist of the following lines in $\mathbb{C}^2$: the vertical lines $x = i$ for $i = 1, \dots, n$, the horizontal lines $y = j$ for $j = 1, \dots, n$, and the diagonal lines $x + y = k$ for $k = 1, \dots, 2n$. Then every point (i, j) with $1 \leq i, j \leq n$ is a triple point, so $\mathcal{L}$ has at least $n^2 = |\mathcal{L}|^2/16$ triple points.

We start with a sketch of the argument for showing that a set $\mathcal{L}$ of lines, with no $|\mathcal{L}|^{1/2}$ lines in a plane, has $O(|\mathcal{L}|^{3/2})$ triple points.

Proof sketch. As for joints, we prune the lines and interpolate to get a surface $Z(F)$ containing all the lines. Suppose there is a polynomial P_F that vanishes on any intersection point of three lines that are contained in $Z(F)$, and that $\deg(P_F) \lesssim \deg(F)$. Since P_F

vanishes on all the triple points, it will also vanish on all the lines thanks to the pruning step, so $\cup\mathcal{L} \subset Z(P_F)$. If $\dim(Z(F) \cap Z(P_F)) = 1$, then we have (using Bézout's Inequality)

$$\deg(\cup\mathcal{L}) \leq \deg(Z(F) \cap Z(P_F)) \leq \deg(F) \cdot \deg(P_F) \lesssim \deg(F)^2.$$

Thus there are $O(\deg(F)^2)$ lines in $\mathcal{L}$. We will use a slightly improved version of interpolation for which $\deg(F) < N^{1/2}$, so that we get a contradiction to the fact that there are N lines in $\mathcal{L}$. On the other hand, if $\dim(Z(F) \cap Z(P_F)) = 2$, then P_F would vanish on a component of $Z(F)$. We will see that then this component must be a plane, and thus by assumption can also not contain many lines. □

We need two things to make this argument work. We need to get an interpolation polynomial that does a little bit better than the usual interpolation. Guth and Katz found a way to do this, making use of the special situation that we are in; they call this "degree reduction". The proof uses random sampling and can be found in Appendix A. To get some feeling for the statement, think of $A \approx C|\mathcal{L}|^{1/2}$ for some large C. Then the lemma gives a polynomial that interpolates the lines and has degree $C^{-1}|\mathcal{L}|^{1/2}$; this is an improvement on the standard line interpolation in Lemma 3.6.

Lemma 3.8 (Degree reduction). *Let $\mathcal{L}$ be a set of lines in $\mathbb{C}^3$ such that each line has at least A distinct intersection points with other lines of $\mathcal{L}$. Then there is $F \in \mathbb{C}[x, y, z]$ with $\cup\mathcal{L} \subset Z(F)$ and $\deg(F) \lesssim A^{-1}|\mathcal{L}|$.*

The second thing we need is that if P_F vanishes on a component of F, then it must be a plane. This does not quite seem to work with a single polynomial, but it turns out that there is a *polynomial vector* that does the job. We think of a polynomial vector T as just a list of polynomials, like the gradient of a polynomial; we can still write $\mathrm{T}(p) = 0$ for each entry equalling zero at p, or $Z(\mathrm{T})$ for the zero set of this list of polynomials. By $\deg(\mathrm{T})$ we mean the maximum degree of the entries of T. The following lemma is proved in Appendix D.

Lemma 3.9 (Triple point polynomial). *For any $F \in \mathbb{C}[x, y, z]$, there exists a polynomial vector $\mathrm{T}_F \in (\mathbb{C}[x, y, z])^3$ such that:*

(*a*) $\deg(\mathrm{T}_F) \leq 3\deg(F)$;

(*b*) *If $p \in Z(F)$ is an intersection point of three lines contained in $Z(F)$, then $\mathrm{T}_F(p) = 0$;*

(*c*) *If $Z(G) \subset Z(F) \cap Z(\mathrm{T}_F)$, then $Z(G)$ contains a plane.*

With these tools we can prove the main theorem of this chapter. It is the case $k = 3$ of Guth-Katz Theorem B.

Theorem 3.10 (Elekes-Kaplan-Sharir). *Let $\mathcal{L}$ be a set of lines in $\mathbb{C}^3$ with no $|\mathcal{L}|^{1/2}$ in a plane. Then*

$$|\mathcal{I}_{\geq 3}(\mathcal{L})| \lesssim |\mathcal{L}|^{3/2}.$$

Proof. Let $N = |\mathcal{L}|$, let I be the set of triple points of $\mathcal{L}$, and assume $|I| = 2CN^{3/2}$. By Lemma E.3, there are $\mathcal{L}' \subset \mathcal{L}$, $I' \subset I$ such that every line $\ell \in \mathcal{L}'$ contains at least $CN^{1/2}$ points of I', and every point of I' is a triple point of $\mathcal{L}'$. Set $N' = |\mathcal{L}'|$. By degree reduction (Lemma 3.8 or Lemma B.3), there is a polynomial F such that $\deg(F) \lesssim C^{-1}N^{-1/2}N'$ and $\cup\mathcal{L}' \subset Z(F)$.[1]

Let T_F be the polynomial vector from Lemma 3.9 (or Theorem D.1). For any triple point $p \in I'$ we have $\mathrm{T}_F(p) = 0$ by Lemma 3.9(b). It follows that for $\ell \in \mathcal{L}'$ we have $|\ell \cap Z(\mathrm{T}_F)| \geq |\ell \cap I'| \geq CN^{1/2}$, while $\deg(\mathrm{T}_F) \leq 3\deg(F) \lesssim C^{-1}N^{-1/2}N'$. For sufficiently large C we have $CN^{1/2}$ larger than $C^{-1}N^{-1/2}N'$, so Lemma A.7 implies $\cup\mathcal{L}' \subset Z(\mathrm{T}_F)$.

We can decompose $Z(F)$ as

$$Z(F) = Z(G) \cup \bigcup_{j=1}^{k} Z(P_j),$$

where $Z(G)$ does not contain a plane and has degree $\deg(G) \leq \deg(F)$, and where each $Z(P_j)$ is a plane, with $k \leq \deg(F)$. Since $Z(G)$ contains no planes, we have $\dim(Z(G) \cap Z(\mathrm{T}_F)) = 1$ by part (c) of Lemma 3.9. Let $\mathcal{L}'_G$ be the subset of lines in $\mathcal{L}'$ that are contained in $Z(G)$. Since T_F vanishes on all lines of $\mathcal{L}'$, we have

$$\deg(\cup\mathcal{L}'_G) \leq \deg(Z(\mathrm{T}_F) \cap Z(G)) \leq \deg(\mathrm{T}_F) \cdot \deg(G) \lesssim (C^{-1}N^{-1/2}N')^2 \lesssim C^{-2}N'.$$

This implies that there are $O(C^{-2}N')$ lines of $\mathcal{L}'$ contained in $Z(G)$.

Now consider a plane P_i. By assumption, it contains at most $N^{1/2}$ lines. Thus the total number of lines on planes is at most

$$\sum_{j=1}^{k} N^{1/2} \leq \deg(F) \cdot N^{1/2} \lesssim C^{-1}N^{-1/2}N' \cdot N^{1/2} \lesssim C^{-1}N'.$$

Altogether we have accounted for only $O(C^{-2}N' + C^{-1}N')$ lines of $\mathcal{L}'$, which for large enough C is less than N', so this is a contradiction. $\square$

Notes. (Note: To be completed.) Theorem 3.5 was first proved by Guth and Katz [17], and later simplified in [28, 23]. We work in $\mathbb{C}^3$ because it is the natural habitat for algebraic geometry, but we observe that the theorem in $\mathbb{R}^3$ directly follows. In fact, with a few changes the proof works in $\mathbb{F}^D$ for any field $\mathbb{F}$, with the bound $O(|\mathcal{L}|^{D/(D-1)})$; see for instance [46, 5].

Theorem 3.10 was first proved by Elekes, Kaplan, and Sharir [10], and another proof can be found in [15, Lecture 15]. The bound in the theorem is tight given the condition that no $|\mathcal{L}|^{1/2}$ lines lie on a plane. If one changes the condition to "no B lines on a plane", for some $B \geq |\mathcal{L}|^{1/2}$, then one can prove the bound $|\mathcal{I}_{\geq 3}(\mathcal{L})| \lesssim B|\mathcal{L}|$ (see [15, Lecture 15]). However, these methods do not seem to give a bound better than $|\mathcal{L}|^{3/2}$ when $B < |\mathcal{L}|^{1/2}$, and it is an open problem whether the bound can be improved in this case. Guth [16] more or less asked the following.

[1] Note that we have to be careful: Although we have moved to a subset of lines of size N', the condition on the number of lines in a plane still only holds in terms of the original N. Fortunately, there is a tradeoff: For smaller N', the condition that a plane contains $N^{1/2}$ lines is weaker, but the property that each line has $CN^{1/2}$ becomes stronger, and these effects cancel each other out.

Problem 3.11. *Let $\mathcal{L}$ be a set of lines in $\mathbb{C}^3$ with no 10 on a plane. Prove that $|\mathcal{I}_{\geq 3}(\mathcal{L})| \lesssim |\mathcal{L}|$.*

Chapter 4

Intersection points of lines in $\mathbb{R}^3$

In this section we will see the proof of the Guth-Katz Theorem A on intersection points (points where at least two of the given lines intersect), which is a step up from the proof of Theorem 3.10 on triple intersection points. A crucial difference is that for intersection points, there is another class of exceptional structures besides planes: doubly-ruled surfaces.

Construction 4.1. Let $S \subset \mathbb{C}^3$ be the doubly-ruled surface defined by $z = xy$. Let $\mathcal{L}_1$ be the set of lines $\{(i, t, it) : t \in \mathbb{C}\}$ for $i \in [n]$, let $\mathcal{L}_2$ be the set of lines $\{(t, j, tj) : t \in \mathbb{C}\}$ for $j \in [n]$, and set $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$. Then every line of $\mathcal{L}_1$ intersects every line of $\mathcal{L}_2$, so we have $|\mathcal{I}_{\geq 2}(\mathcal{L})| = \Theta(|\mathcal{L}|^2)$. On the other hand, any plane contains at most two lines of $\mathcal{L}$, so this construction is unrelated to Construction 3.7.

In Section 4.1 we will briefly introduce ruled surfaces (doubly-ruled and singly-ruled), and we will prove that a similar construction works on any doubly-ruled surface, but not on a singly-ruled surface. A more detailed introduction to ruled surfaces can be found in Section C.2, along with proofs of the basic facts about ruled surfaces that we use in Section 4.1.

As in Chapter 3, we will use a polynomial that "detects" certain points on the surface, and we will have to deal with the case where this polynomial vanishes on the entire surface. Unfortunately, unlike in Lemma 3.9, where this only happened for the expected exception (a plane), in this case the polynomial vanishes on any ruled surface. Here the expected exceptions are only the doubly-ruled surfaces (and planes), but the detection polynomial does not quite distinguish these from singly-ruled surfaces. Thus we will have to analyze lines on singly-ruled surfaces, and we will see that for them the intended bound on the number of intersection points does hold. Once that is established, the analysis is much like that for Theorem 3.10, with some new twists in the argument.

In this chapter we will write $I(\mathcal{L})$ instead of $\mathcal{I}_{\geq 2}(\mathcal{L})$ for the set of points where at least two lines from $\mathcal{L}$ intersect.

4.1 Ruled surfaces

Let us start with the definition of ruled surfaces, and of the two main subclasses (aside from planes, which can be considered "infinitely-ruled").

Definition 4.2 (Ruled surfaces)**.** Note that we only consider irreducible surfaces.

- **Ruled surface**: An irreducible surface $S \subset \mathbb{C}^3$ is a *ruled surface* if every point of S lies on a line that is contained in S;
- **Doubly-ruled surface**: A ruled surface $S \subset \mathbb{C}^3$ is *doubly-ruled* if every point of S lies on exactly two lines that are contained in S;
- **Singly-ruled surface**: A ruled surface $S \subset \mathbb{C}^3$ is *singly-ruled* if there is a curve $C \subset S$ such that every point in $S \backslash C$ lies on exactly one line contained in S.

Various properties of ruled surfaces are proved in Section C.2, but the most important fact to have in mind is the following, proved as Lemma C.4.

Lemma 4.3. *Every ruled surface is either singly-ruled, doubly-ruled, or it is a plane.*

As explained in the introduction to this section, we have to bound intersection points of lines on ruled surfaces, and we do this in the following theorem. The proof uses some of the properties of ruled surfaces established in Section C.2. In the proof Theorem 4.6, we will apply this to a surface of degree $|\mathcal{L}|^{1/2}$, and in that case we get the familiar bound $O(|\mathcal{L}|^{3/2})$.

Theorem 4.4. *Let $S \subset \mathbb{C}^3$ be a singly-ruled surface. If $\mathcal{L}$ is a set of lines contained in S, then*

$$|I(\mathcal{L})| \lesssim \deg(S) \cdot |\mathcal{L}|.$$

Proof. If S is a cone (see Section C.2 for the definition), then the lines contained in it have only one intersection point (the apex), and we are done. Thus we can assume that S is not a cone, so by Lemma C.2 S has no exceptional point (a point that lies in infinitely many lines contained in S). Moreover, by Lemma C.6, S contains at most two exceptional lines, i.e., lines that contain infinitely many intersection points with other lines contained in S. Hence such lines can together give at most $2|\mathcal{L}|$ intersection points of $\mathcal{L}$. In the remainder of the proof we only have to consider non-exceptional points and non-exceptional lines.

It suffices to prove that each line $\ell \in \mathcal{L}$ contains $O(\deg(S))$ intersection points. Fix $\ell \in \mathcal{L}$. Take any plane π containing ℓ. Then $\pi \cap S$ is a curve of degree at most $\deg(S)$ containing ℓ; let C be this curve with ℓ removed. If another line ℓ' on S is contained in π, then it is contained in C, so there are at most $\deg(S)$ such lines, and at most $\deg(S)$ intersection points on ℓ coming from lines contained in π.

Let ℓ' be a line on S that intersects ℓ but is not contained in π. The intersection point $p \in \ell \cap \ell'$ must be isolated on ℓ, in the sense that there is an open neighborhood B of p such that p is the only intersection point of ℓ in B; otherwise ℓ would be exceptional. However, by Lemma C.8, for an open neighborhood $p \in B' \subset B$, there is a line ℓ'' on S that intersects $B' \cap \pi$ in a point other than p. Therefore, these lines must intersect π in C in points arbitrarily close to p, which is only possible if $p \in C \cap \ell$, because C is a continuous curve.

Thus every intersection point of ℓ with another line ℓ' on S must lie in $C \cap \ell$. By Bézout's inequality (Theorem A.4), we have $|C \cap \ell| \leq \deg(S)$, so there are at most $\deg(S)$ intersection points on ℓ. This completes the proof. $\square$

4.2 Proof of Guth-Katz Theorem A

The following lemma is proved in Appendix D.

Lemma 4.5 (Flecnode polynomial). *For any $F \in \mathbb{C}[x, y, z]$, there exists a polynomial $\mathrm{Fl}_F \in \mathbb{C}[x, y, z]$ such that:*

a) $\deg(\mathrm{Fl}_F) \lesssim \deg(F)$*;*

b) If $p \in Z(F)$ is contained in a line that is contained in $Z(F)$, then $\mathrm{Fl}_F(p) = 0$;

c) If F is irreducible and $Z(F) \subset Z(\mathrm{Fl}_F)$, then $Z(F)$ is a ruled surface.

We are now prepared to prove Guth-Katz Theorem A.

Theorem 4.6. *Let $\mathcal{L}$ be a set of lines in $\mathbb{C}^3$ such that no $|\mathcal{L}|^{1/2}$ are in a plane or doubly-ruled surface. Then*

$$|I(\mathcal{L})| \lesssim |\mathcal{L}|^{3/2}.$$

Proof. Let $N = |\mathcal{L}|$ and assume that $|I(\mathcal{L})| = 2CN^{3/2}$. By pruning (Lemma E.2), there is $\mathcal{L}' \subset \mathcal{L}$ such that if we set $I' = I(\mathcal{L}')$, then $|I'| \geq CN^{3/2}$ and for all $\ell \in \mathcal{L}'$ we have $|\ell \cap I'| \geq CN^{1/2}$. By degree reduction (Lemma B.3), there is a polynomial F such that $\deg(F) \lesssim C^{-1}N^{1/2}$ and $\cup\mathcal{L}' \subset Z(F)$.

First we decompose $Z(F)$ into its irreducible components, as $Z(F) = \cup_{i=1}^{k} Z(F_i)$, with F_i irreducible, and $\sum_{i=1}^{k} \deg(F_i) \leq \deg(F)$. Write $\mathcal{L}_i'$ for the subset of lines in $\mathcal{L}'$ that are contained in $Z(F_i)$.

Call an intersection point *internal* if it is an intersection point of two lines in the same $\mathcal{L}_i'$, and call all other intersection points *external*. A line in $\mathcal{L}'$ intersects the components of $Z(F)$ that do not contain it in at most $\deg(F)$ points; this is a bound on the number of external intersection points on a line. Hence the total number of external intersection points is bounded by

$$|\mathcal{L}'| \cdot \deg(F) \lesssim N \cdot (C^{-1}N^{1/2}) \leq C^{-1}N^{3/2}.$$

If a surface $Z(F_i)$ is a plane or doubly-ruled surface, then by assumption we have $|\mathcal{L}_i'| \leq N^{1/2}$, and therefore $|I(\mathcal{L}_i')| \leq N$. Hence the total number of internal intersection points on these components is at most

$$\sum_{i=1}^{k} N \leq \deg(F) \cdot N \lesssim C^{-1}N^{3/2}.$$

For a surface $Z(F_i)$ that is singly-ruled we have $|I(\mathcal{L}_i')| \lesssim \deg(F_i)|\mathcal{L}_i'|$ by Lemma 4.4, so the number of internal intersection points on such components is at most

$$\sum_{i=1}^{k} \deg(F_i) \cdot |\mathcal{L}_i'| \leq |\mathcal{L}'| \cdot \sum_{i=1}^{k} \deg(F_i) \leq N \cdot \deg(F) \lesssim C^{-1}N^{3/2}.$$

Now consider a surface $Z(F_i)$ that is not ruled. Let Fl_{F_i} be the flecnode polynomial from Lemma 4.5. By Lemma 4.5(b), Fl_{F_i} vanishes on every line contained in $Z(F_i)$, so in particular $\cup\mathcal{L}'_i \subset Z(\mathrm{Fl}_{F_i})$. By Lemma 4.5(c), the fact that $Z(F_i)$ is not ruled implies that $\dim(Z(F_i) \cap Z(\mathrm{Fl}_{F_i})) = 1$. Thus we get, using Theorem A.5,

$$\deg(\cup\mathcal{L}'_i) \leq \deg(Z(F_i) \cap Z(\mathrm{Fl}_{F_i})) \leq \deg(F_i) \cdot \deg(\mathrm{Fl}_{F_i}) \lesssim \deg(F_i)^2.$$

Therefore, the number of lines contained in non-ruled components is at most

$$\sum_i \deg(F_i)^2 \leq \deg(F)^2 \lesssim C^{-2}N.$$

Let $\mathcal{L}^*$ be the set of these lines.

We now have a small subset $\mathcal{L}^*$ of $\mathcal{L}$, with $|\mathcal{L}^*| \lesssim C^{-2}N$, which must account for almost all the intersection points, namely

$$|I(\mathcal{L}^*)| \gtrsim CN^{3/2} - 3C^{-1}N^{3/2} \gtrsim CN^{3/2}.$$

We can assume that the N we started with is minimal with the property that there exists a set of N lines with at least $2CN^{3/2}$ intersection points, but no $N^{1/2}$ on a plane or doubly-ruled surface. Then we almost have a contradiction, because we have a set of fewer than N lines with this property, except for the fact that $\mathcal{L}^*$ has no $|\mathcal{L}|^{1/2}$ lines in a plane or doubly-ruled surface, instead of $|\mathcal{L}^*|^{1/2}$.

We can deal with this as follows. It suffices to show that for some $M < N$ there is a set of at most M lines with no $M^{1/2}$ lines in a plane or doubly-ruled surface, but still at least $2CM^{3/2}$ intersection points; we will show this for $M = C^{-2}N$. Set $\mathcal{M} = \mathcal{L}^*$. If a set of at least $C^{-1}N^{1/2}$ lines of $\mathcal{M}$ lies on a common plane or doubly-ruled surface, then we remove these lines from $\mathcal{M}$. We repeat this until $\mathcal{M}$ has no $C^{-1}N^{1/2}$ lines in a plane or doubly-ruled surface; we'll see below that $\mathcal{M}$ cannot be empty. Set $\mathcal{N} = \mathcal{L}^* \backslash \mathcal{M}$.

We can partition $\mathcal{N} = \cup_{i=1}^k \mathcal{N}_i$ so that each $\mathcal{N}_i$ consists of lines lying on a single plane or doubly-ruled surface. We can do this with $C^{-1}N^{1/2} \leq |\mathcal{N}_i| < N^{1/2}$ for each i, and we have $k \leq |\mathcal{L}^*|/(C^{-1}N^{1/2}) \lesssim C^{-1}N^{1/2}$. It follows that the total number of internal intersection points of the $\mathcal{N}_i$ is at most $k \cdot |\mathcal{N}_i|^2 \lesssim C^{-1}N^{3/2}$. Any line in $\mathcal{L}^*$ has at most two intersection points with each of the planes and doubly-ruled surfaces that the lines of $\mathcal{N}$ lie on, but that the line itself does not lie on. Thus the total number of external intersection points between different $\mathcal{N}_i$ is at most $\sum |\mathcal{N}_i| \cdot 2k \lesssim C^{-1}N^{3/2}$. Similarly, the number of external intersection points between $\mathcal{M}$ and $\mathcal{N}$ is at most $|\mathcal{M}| \cdot 2k \lesssim C^{-1}N^{3/2}$.

Set $M = C^{-2}N$. Then $\mathcal{M}$ has at most $|\mathcal{L}^*| \lesssim M$ lines, and by construction it has no $C^{-1}N^{1/2} = M^{1/2}$ lines on a plane or doubly-ruled surface. On the other hand, we must still have $|I(\mathcal{M})| \gtrsim CN^{3/2} - 3C^{-1}N^{3/2} \gtrsim CN^{3/2} \gtrsim C^4M^{3/2}$, which for sufficiently large C is more than $2CM^{3/2}$. This contradicts the minimality of N. $\square$

Chapter 5

Incidences in $\mathbb{R}^2$

In this chapter we make a detour from incidences in $\mathbb{R}^3$ to incidences in $\mathbb{R}^2$. To prove the Guth-Katz Theorem B, we need a technique introduced by Guth and Katz called "polynomial partitioning". To introduce this technique, we use it to prove some point-curve incidence theorems in $\mathbb{R}^2$ (which had been proved with other methods before polynomial partitioning). First we prove the Szemerédi-Trotter Theorem, which is the prototype for all later incidence theorems. Then we prove the Pach-Sharir Theorem, which is a generalization to algebraic curves, and see some interesting applications, including the unit distance bound (Theorem 5.9) and some bounds on distinct distances that predate the bound of Guth-Katz (Theorem 5.13).

The polynomial partitioning technique is based on the following theorem. We prove it in Appendix B, along with its higher-dimensional versions.

Theorem 5.1 (Polynomial partitioning in the plane). *Let $\mathcal{P} \subset \mathbb{R}^2$. For any integer $r \geq 1$ there exists a polynomial $F \in \mathbb{R}[x, y]$ of degree r such that $\mathbb{R}^2 \backslash Z(F)$ has $O(r^2)$ connected components, each containing $O(|\mathcal{P}|/r^2)$ points of $\mathcal{P}$.*

5.1 The Szemerédi-Trotter Theorem

Recall that given $\mathcal{P} \subset \mathbb{R}^2$ and a set $\mathcal{S}$ of subsets of $\mathbb{R}^2$, we define the set of *incidences* to be $I(\mathcal{P}, \mathcal{S}) := \{(p, s) \in \mathcal{P} \times \mathcal{S} : p \in s\}$. For clarity, we prove here a slightly simplified version of the Szemerédi-Trotter Theorem, where we assume that the point set and line set have the same cardinality.

Theorem 5.2 (Balanced Szemerédi-Trotter). *Let $\mathcal{P}$ be a set of at most n points in $\mathbb{R}^2$ and $\mathcal{L}$ a set of at most n lines in $\mathbb{R}^2$. Then*

$$|I(\mathcal{P}, \mathcal{L})| \lesssim n^{4/3}.$$

Proof. Let F be a partitioning polynomial for $\mathcal{P}$ as in Theorem 5.1, of degree r, for an integer r that will be chosen at the end of the proof. Set $\mathcal{P}_F := \mathcal{P} \cap Z(F)$, and let $\mathcal{L}_F \subset \mathcal{L}$ be the subset of lines that are contained in $Z(F)$.

There are at most r lines in $\mathcal{L}_F$, so we have

$$|I(\mathcal{P}, \mathcal{L}_F)| \leq r \cdot |\mathcal{P}| \leq rn.$$

By Bézout's Inequality (Theorem A.4), a line that is not contained in $Z(F)$ intersects it in at most r points, so

$$|I(\mathcal{P}_F, \mathcal{L}\backslash\mathcal{L}_F)| \leq r \cdot |\mathcal{L}| \leq rn.$$

Let $\mathcal{I}_1 \subset I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{L}\backslash\mathcal{L}_F)$ be the subset of incidences (p, ℓ) such that p is the only incidence on ℓ in the cell that p lies in. Let $\mathcal{I}_2 \subset I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{L}\backslash\mathcal{L}_F)$ be the subset of incidences (p, ℓ) such that ℓ has at least two incidences in the cell that p lies in. We have $|I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{L}\backslash\mathcal{L}_F)| = |\mathcal{I}_1| + |\mathcal{I}_2|$.

A line in $\mathcal{L}\backslash\mathcal{L}_F$ intersects at most $r+1$ cells. Thus we can bound $|\mathcal{I}_1|$ by

$$|\mathcal{I}_1| \leq (r+1) \cdot |\mathcal{L}| \lesssim rn.$$

To bound $|\mathcal{I}_2|$, we observe that any pair of points is contained in only one line, and this results in at most two incidences. Therefore, any set of k points in a cell contributes at most $2\binom{k}{2}$ incidences to $\mathcal{I}_2$. Applying this to the $O(n/r^2)$ points in a connected component of $\mathbb{R}^2\backslash Z(F)$, and summing over all $O(r^2)$ connected components, we get

$$|\mathcal{I}_2| \lesssim r^2 \cdot 2\binom{n/r^2}{2} \lesssim \frac{n^2}{r^2}.$$

Combinining all of the above we have

$$|I(\mathcal{P}, \mathcal{L})| = |I(\mathcal{P}, \mathcal{L}_F)| + |I(\mathcal{P}_F, \mathcal{L}\backslash\mathcal{L}_F)| + |I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{L}\backslash\mathcal{L}_F)| \lesssim \frac{n^2}{r^2} + r \cdot n.$$

We can minimize the right-hand side by choosing $r = n^{1/3}$, which results in the bound in the theorem. $\square$

Here is a nice application. Compare it with the unit distances problem (see Theorem 5.9 or the Introduction), where we ask how many pairs of points can determine a line segment of the same length. A natural next step is to ask how many *triples* of points can determine a *triangle* with the same area. Just like for unit distances, there is an interesting bound, but we are not yet able to match the best known lower bound, which in this case is $\Omega(|\mathcal{P}|^2 \log |\mathcal{P}|)$.

Corollary 5.3. *A finite point set $\mathcal{P} \subset \mathbb{R}^2$ determines $O(|\mathcal{P}|^{7/3})$ unit-area triangles.*

Proof. We show that a fixed point $p \in \mathcal{P}$ is involved in $O(|\mathcal{P}|^{4/3})$ unit-area triangles, which implies the statement. For each other point $q \in \mathcal{P}$, let ℓ_q be the line to the left of the line pq at distance $2/D(p, q)$. For every point $r \in \ell_q$, the triangle pqr has unit area. By Theorem 5.4, the point set $\mathcal{P}$ and the set of lines ℓ_q have $O(|\mathcal{P}|^{4/3})$ incidences. This means that there are $O(|\mathcal{P}|^{4/3})$ unit-area triangles involving p (note that for every triangle pqr, either r is to the left of pq or q is to the left of pr). $\square$

The proof of the balanced Szemerédi-Trotter Theorem can easily be extend to give the full "unbalanced" version, but for clarity we have not done this here. We state it now anyway; it follows from Theorem 5.8 below.

Theorem 5.4 (Szemerédi-Trotter). *Let $\mathcal{P} \subset \mathbb{R}^2$ and let $\mathcal{L}$ be a set of lines in $\mathbb{R}^2$. Then*

$$|I(\mathcal{P}, \mathcal{L})| \lesssim |\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + |\mathcal{P}| + |\mathcal{L}|.$$

The following is a useful corollary.

Corollary 5.5. *Let $\mathcal{P} \subset \mathbb{R}^2$ and $k \geq 2$. The number of k-rich lines of $\mathcal{P}$ (lines containing at least k points of $\mathcal{P}$) is*

$$O\left(\frac{|\mathcal{P}|^2}{k^3} + \frac{|\mathcal{P}|}{k}\right).$$

The number of incidences between $\mathcal{P}$ and k-rich lines of $\mathcal{P}$ is

$$O\left(\frac{|\mathcal{P}|^2}{k^2} + |\mathcal{P}|\right).$$

Proof. Let $\mathcal{L}$ be the set of k-rich lines. Then we have $k|\mathcal{L}| \lesssim |\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + |\mathcal{P}|$, so we have $|\mathcal{L}|^{1/3} \lesssim |\mathcal{P}|^{2/3}/k$ or otherwise $|\mathcal{L}| \lesssim |\mathcal{P}|/k$. Plugging these bounds back into the Szemerédi-Trotter bound gives

$$|I(\mathcal{P},\mathcal{L})| \lesssim |\mathcal{P}|^{2/3}\left(\frac{|\mathcal{P}|^2}{k^3}\right)^{2/3} + |\mathcal{P}|^{2/3}\left(\frac{|\mathcal{P}|}{k}\right)^{2/3} + |\mathcal{P}| + \frac{|\mathcal{P}|^2}{k^3} + \frac{|\mathcal{P}|}{k}.$$

This gives the second bound in the corollary. Note that we have $|\mathcal{P}|^{4/3}/k^{2/3} \leq |\mathcal{P}|^2/k^2$ if $k \leq |\mathcal{P}|^{1/2}$, and $|\mathcal{P}|^{4/3}/k^{2/3} \leq |\mathcal{P}|$ if $k \geq |\mathcal{P}|^{1/2}$. $\square$

5.2 The Pach-Sharir Theorem

We now prove a generalization of Theorem 5.4 to algebraic curves, which will turn out to have many applications (see also Chapter ???). First we need two lemmas, which give weaker versions of the main bound.

Lemma 5.6. *Given a set $\mathcal{P}$ of points and a set $\mathcal{C}$ of curves such that any s points of $\mathcal{P}$ are contained in at most t curves of $\mathcal{C}$, we have*

$$|I(\mathcal{P},\mathcal{C})| \lesssim_{s,t} |\mathcal{P}|^s + |\mathcal{C}|.$$

Proof. Let $\mathcal{I}_1$ be the subset of incidences $(p, C) \in I(\mathcal{P},\mathcal{C})$ such that C has at most $s-1$ incidences, and let $\mathcal{I}_2$ be the subset of those for which C has at least s incidences. Clearly $|\mathcal{I}_1| \leq (s-1)\cdot|\mathcal{C}|$. On the other hand, the fact that any s points are contained in at most t curves implies that s points can give at most st incidences from $\mathcal{I}_2$, so $|\mathcal{I}_2| \leq st\binom{|\mathcal{P}|}{s}$. $\square$

Lemma 5.7. *Given a set $\mathcal{P}$ of points in $\mathbb{R}^2$ and a set $\mathcal{C}$ of irreducible algebraic curves of degree at most d in $\mathbb{R}^2$, we have*

$$|I(\mathcal{P},\mathcal{C})| \lesssim_d |\mathcal{P}| + |\mathcal{C}|^2.$$

Proof. Let $\mathcal{I}_1$ be the set of incidences $(p, C) \in I(\mathcal{P},\mathcal{C})$ such that p is contained in at most one curve of $\mathcal{C}$, and let $\mathcal{I}_2$ be those for which p is contained in at least two curves of $\mathcal{C}$. Then $|\mathcal{I}_1| \leq |\mathcal{P}|$. By Bézout's Inequality (Theorem A.4), any two curves of $\mathcal{C}$ share at most d^2 points. Thus two curves give at most $2d^2$ incidences from $\mathcal{I}_2$, so $|\mathcal{I}_2| \leq d^2\binom{|\mathcal{C}|}{2}$. $\square$

Theorem 5.8 (Pach-Sharir). *Let $\mathcal{P} \subset \mathbb{R}^2$ and let $\mathcal{C}$ be a set of algebraic curves in $\mathbb{R}^2$ of degree d. Suppose that any s points of $\mathcal{P}$ are contained in at most t curves of $\mathcal{C}$. Then*

$$|I(\mathcal{P}, \mathcal{C})| \lesssim_{d,s,t} |\mathcal{P}|^{\frac{s}{2s-1}}|\mathcal{C}|^{\frac{2s-2}{2s-1}} + |\mathcal{P}| + |\mathcal{C}|.$$

Proof. We can assume that all curves are irreducible, because if they are not, we can split them into their irreducible components. Let $\mathcal{C}'$ be the set of all irreducible components of all curves in $\mathcal{C}$, so $|\mathcal{C}'| \le d|\mathcal{C}|$. A curve in $\mathcal{C}'$ may occur multiple times as a component of a curve in $\mathcal{C}$, but we can deal with that as follows. The total number of incidences from curves with less than s incidences is at most $d|\mathcal{C}| \cdot (s-1)$. On the other hand, if a curve of $\mathcal{C}'$ has at least s incidences, then the curve can occur at most t times as an irreducible component of a curve in $\mathcal{C}$. This tells us that

$$|I(\mathcal{P}, \mathcal{C})| \le t \cdot |I(\mathcal{P}, \mathcal{C}')| + ds|\mathcal{C}|. \tag{5.1}$$

Let F be a partitioning polynomial for $\mathcal{P}$ of degree r as in Theorem B.6. Set $\mathcal{P}_F := \mathcal{P} \cap Z(F)$, and let $\mathcal{C}'_F \subset \mathcal{C}'$ be the subset of curves that are contained in $Z(F)$. We will refer to the connected components of $\mathbb{R}^2 \backslash Z(F)$ as "cells".

There are at most r curves contained in $\mathcal{C}'_F$, so using Lemma 5.7 we have

$$|I(\mathcal{P}, \mathcal{C}'_F)| \lesssim_d |\mathcal{P}| + r^2.$$

By Bézout's Inequality (Theorem A.4), an irreducible curve that is not contained in $Z(F)$ intersects it in at most dr points, so we have

$$|I(\mathcal{P}_F, \mathcal{C}' \backslash \mathcal{C}'_F)| \lesssim_d r \cdot |\mathcal{C}'|.$$

Let $\mathcal{I}_1 \subset I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{C}\backslash\mathcal{C}_F)$ be the subset of incidences (p, C) such that C has at most $s-1$ incidences in the cell that p lies in. Let $\mathcal{I}_2 \subset I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{C}\backslash\mathcal{C}_F)$ be the subset of incidences (p, C) such that C has at least s incidences in the cell that p lies in. We have $|I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{C}\backslash\mathcal{C}_F)| = |\mathcal{I}_1| + |\mathcal{I}_2|$.

For $C \in \mathcal{C}$ we have by Theorem A.12 that $C \backslash Z(F)$ has $O_d(r)$ connected components, which means that C intersects $O_d(r)$ cells. This implies

$$|\mathcal{I}_1| \le (s-1) \cdot O_d(r) \cdot |\mathcal{C}'| \lesssim_{d,s} r|\mathcal{C}'|.$$

To bound $|\mathcal{I}_2|$, we use the assumption that any s points of $\mathcal{P}$ are contained in at most t curves of $\mathcal{C}'$. Thus every s points in a cell together contribute to at most st incidences from $\mathcal{I}_2$. Therefore, if a cell contains k points, then at most $st\binom{k}{s}$ incidences from $\mathcal{I}_2$ occur in that cell. Applying this to the $O(|\mathcal{P}|/r^2)$ points in a cell, and summing over all cells, we get

$$|\mathcal{I}_2| \le O(r^2) \cdot st \binom{|\mathcal{P}|/r^2}{s} \lesssim_{s,t} \frac{|\mathcal{P}|^s}{r^{2s-2}}.$$

Combining all of the above we have

$$|I(\mathcal{P}, \mathcal{C}')| = |I(\mathcal{P}, \mathcal{C}'_F)| + |I(\mathcal{P}_F, \mathcal{C}' \backslash \mathcal{C}'_F)| + |I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{C}' \backslash \mathcal{C}'_F)| \lesssim_{d,s,t} \frac{|\mathcal{P}|^s}{r^{2s-2}} + r|\mathcal{C}'| + |\mathcal{P}| + r^2.$$

Going back to the set $\mathcal{C}$ using (5.1) and $|\mathcal{C}'| \leq |\mathcal{C}|$, we get

$$|I(\mathcal{P},\mathcal{C})| \lesssim_{d,s,t} \frac{|\mathcal{P}|^s}{r^{2s-2}} + r|\mathcal{C}| + |\mathcal{P}| + r^2 + |\mathcal{C}|.$$

We can minimize the first two terms on the right by choosing $r = |\mathcal{P}|^{s/(2s-1)}|\mathcal{C}|^{-1/(2s-1)}$, which gives the main term in the bound to prove. For that choice of r, the assumption $|\mathcal{C}| \geq |\mathcal{P}|^{1/2}$ gives

$$r^2 \lesssim |\mathcal{P}|^{\frac{2s}{2s-1}}|\mathcal{C}|^{-\frac{2}{2s-1}} \lesssim |\mathcal{P}|^{\frac{2s}{2s-1}}|\mathcal{P}|^{-\frac{1}{2s-1}} = |\mathcal{P}|,$$

which shows that we have the bound in the theorem.

We do still have to check that our choice of r satisfies the requirement of Theorem 5.1 that $r \geq 1$. This holds if $|\mathcal{P}| \geq |\mathcal{C}|^{1/s}$. Otherwise, if $|\mathcal{P}|^s < |\mathcal{C}|$, then Lemma 5.6 gives $|I(\mathcal{P},\mathcal{C})| \lesssim_{s,t} |\mathcal{C}|$, which is accounted for in the bound of the theorem. □

Theorem 5.9 (Unit distances). *For any finite $\mathcal{P} \subset \mathbb{R}^2$, the number $|U_2(\mathcal{P})|$ of unit distances satisfies*

$$|U_2(\mathcal{P})| \lesssim |\mathcal{P}|^{4/3}$$

Proof. Let $\mathcal{C}$ be the set of $|\mathcal{P}|$ unit circles centered at the points of $\mathcal{P}$. Then we have $2|U_2(\mathcal{P})| = |I(\mathcal{P},\mathcal{C})|$, since for every two points at unit distance, there are two incidences (the unit circle around each point hits the other point). By basic geometry, any two points are contained in at most two unit circles, so Theorem 5.8 gives $|I(\mathcal{P},\mathcal{C})| \lesssim |\mathcal{P}|^{4/3}$. □

5.3 Distinct distances pre-Guth-Katz

Corollary 5.10. *For any finite $\mathcal{P} \subset \mathbb{R}^2$, the number $|D(\mathcal{P})|$ of distinct distances satisfies*

$$|D(\mathcal{P})| \gtrsim |\mathcal{P}|^{2/3}.$$

Proof. By Theorem 5.9, any distance occurs at most $|\mathcal{P}|^{4/3}$ times. This implies that there must be $\Omega(|\mathcal{P}|^2/|\mathcal{P}|^{4/3}) = \Omega(|\mathcal{P}|^{2/3})$ distinct distances. □

Corollary 5.11. *For any finite $\mathcal{P} \subset \mathbb{R}^2$ we have*

$$|D(\mathcal{P})| \gtrsim |\mathcal{P}|^{3/4}.$$

Proof. Let Δ be the number of distinct distances. Around every point of $\mathcal{P}$, we can draw Δ circles (or less) that cover all other points; let $\mathcal{C}$ be the set of these circles, so $|\mathcal{C}| \leq \Delta \cdot |\mathcal{P}|$. By construction we have $|I(\mathcal{P},\mathcal{C})| \approx |\mathcal{P}|^2$. By basic geometry, any three points are contained in at most one circle. Applying Theorem 5.8 with $s = 3$ and $t = 1$, we get

$$|I(\mathcal{P},\mathcal{C})| \lesssim |\mathcal{P}|^{3/5}(\Delta|\mathcal{P}|)^{4/5} + |\mathcal{P}| + \Delta|\mathcal{P}| \lesssim \Delta^{4/5}|\mathcal{P}|^{7/5}.$$

Thus we have $\Delta^{4/5}|\mathcal{P}|^{7/5} \gtrsim |\mathcal{P}|^2$, which implies $\Delta \gtrsim |\mathcal{P}|^{3/4}$. □

Lemma 5.12. *Let $\mathcal{P} \subset \mathbb{R}^2$ and let $\mathcal{C}$ be a set of irreducible algebraic curves in $\mathbb{R}^2$ of degree d. Suppose that any two points of $\mathcal{P}$ are* adjacent *on at most t curves of $\mathcal{C}$. Then*

$$|I(\mathcal{P},\mathcal{C})| \lesssim_d t^{1/3}|\mathcal{P}|^{2/3}|\mathcal{C}|^{2/3} + |\mathcal{P}| + t|\mathcal{C}|.$$

Proof. The proof is very similar to that of Theorem 5.8, so we focus on the differences. Let F be a partitioning polynomial for $\mathcal{P}$ of degree r and define $\mathcal{P}_F$ and $\mathcal{C}_F$ as before. We have $|I(\mathcal{P},\mathcal{C}_F)| \lesssim_d |\mathcal{P}| + r^2$ and $|I(\mathcal{P}_F,\mathcal{C}\backslash\mathcal{C}_F)| \lesssim_d r \cdot |\mathcal{C}|$.

We now split up all curves of $\mathcal{C}$ by cutting out $Z(F)$ and separating all connected components of the remainder. More precisely, for $C \in \mathcal{C}$ we have by Theorem A.12 that $C\backslash Z(F)$ has $O_d(r)$ connected components. Let $\mathcal{C}^*$ be the set of all these connected components; we have $|\mathcal{C}^*| \lesssim_d r|\mathcal{C}|$ and $|I(\mathcal{P}\backslash\mathcal{P}_F,\mathcal{C}^*)| = |I(\mathcal{P}\backslash\mathcal{P}_F,\mathcal{C}\backslash\mathcal{C}_F)|$. Every object in $\mathcal{C}^*$ lies in a single cell and forms a continuous curve.

Let $\mathcal{I}_1 \subset I(\mathcal{P}\backslash\mathcal{P}_F,\mathcal{C}^*)$ be the subset of incidences (p, C) such that C has at most one incidence. Let $\mathcal{I}_2$ be the subset of incidences (p, C) such that C has at least two incidences; since C is connected, at least two of these incidences are adjacent, and since C lies in a single cell, so do these incidences. We have $|I(\mathcal{P}\backslash\mathcal{P}_F,\mathcal{C}^*)| = |\mathcal{I}_1| + |\mathcal{I}_2|$.

We can bound $|\mathcal{I}_1|$ by
$$|\mathcal{I}_1| \leq |\mathcal{C}^*| \lesssim_d r|\mathcal{C}|.$$
To bound $|\mathcal{I}_2|$, we use the assumption that any two points of $\mathcal{P}$ are adjacent on at most t curves of $\mathcal{C}$. As we observed, if an incidence (p, C) is included in $\mathcal{I}_2$, then there are at least two adjacent incidences on C in the corresponding cell. Thus every two points in a cell together contribute to at most $2t$ incidences from $\mathcal{I}_2$. This gives
$$|\mathcal{I}_2| \leq O(r^2) \cdot 2t\binom{|\mathcal{P}|/r^2}{2} \lesssim t\frac{|\mathcal{P}|^2}{r^2}.$$
Choosing $r = t^{1/3}|\mathcal{P}|^{2/3}|\mathcal{C}|^{-1/3}$ gives the bound in the lemma, as long as $r \geq 1$. Otherwise we have $|\mathcal{P}|^2 < |\mathcal{C}|/t$, so Lemma 5.6 gives (the dependence on t in Lemma 5.6 follows easily from its proof)
$$|I(\mathcal{P},\mathcal{C})| \lesssim t|\mathcal{P}|^2 + |\mathcal{C}| \lesssim |\mathcal{C}|.$$
This finishes the proof. □

Theorem 5.13 (Székely). *For any finite $\mathcal{P} \subset \mathbb{R}^2$ we have*
$$|D(\mathcal{P})| \gtrsim |\mathcal{P}|^{4/5}.$$

Proof. Again let Δ be the number of distinct distances, around every point draw Δ circles covering all other points, and let $\mathcal{C}$ be the set of these circles. We have $|\mathcal{C}| = \Delta|\mathcal{P}|$ and $|I(\mathcal{P},\mathcal{C})| \approx |\mathcal{P}|^2$.

Let t be an integer that will be chosen at the end of the proof. Let $\mathcal{I}_1$ be the set of incidences $(p, C) \in I(\mathcal{P},\mathcal{C})$ for which there is an incidence (q, C) such that p and q are adjacently contained in more than t circles. Let $\mathcal{I}_2$ be the set of remaining incidences, so any two points are adjacent incidences of $\mathcal{I}_2$ on at most t circles.

By Lemma 5.12 we have
$$|\mathcal{I}_2| \lesssim t^{1/3}|\mathcal{P}|^{2/3}(\Delta|\mathcal{P}|)^{2/3} + \Delta|\mathcal{P}| \lesssim t^{1/3}\Delta^{2/3}|\mathcal{P}|^{4/3} + \Delta|\mathcal{P}|.$$
On the other hand, we can separately bound the incidences in $\mathcal{I}_1$, using the observation that if two points have many circles passing through both, then these circles have their centers on the bisector of these two points (the line that has the same distance to both

points). By Corollary 5.5, there are at most $|\mathcal{P}|^2/t^2 + |\mathcal{P}|$ incidences of points of $\mathcal{P}$ with t-rich lines. For one such incidence (p, ℓ), there are Δ circles around the point, each of which has at most four adjacent point-circle incidences between two points whose bisector is ℓ (there are at most two such pairs of points, each giving two incidences). Thus we get

$$|\mathcal{I}_1| \lesssim \frac{\Delta|\mathcal{P}|^2}{t^2} + \Delta|\mathcal{P}|.$$

We choose $t = \Delta^{1/7}|\mathcal{P}|^{2/7}$, so that

$$|I(\mathcal{P}, \mathcal{C})| = |\mathcal{I}_1| + |\mathcal{I}_2| \lesssim \Delta^{5/7}|\mathcal{P}|^{10/7} + k|\mathcal{P}|.$$

Thus we have

$$\Delta^{5/7}|\mathcal{P}|^{10/7} \gtrsim |\mathcal{P}|^2,$$

which implies $\Delta \gtrsim |\mathcal{P}|^{4/5}$. $\square$

Chapter 6

Point-line incidences in $\mathbb{R}^3$

In this chapter we prove Guth-Katz Theorem B, which bounds the number of k-rich intersection points of a set of lines. We will do this indirectly, by first proving an incidence theorem for points and lines, similar to the Szemerédi-Trotter Theorem. Theorem B will then follow from this theorem, together with Theorem 3.10 on triple intersection points.

6.1 The statement

Here is the "three-dimensional Szemerédi-Trotter Theorem" due to Guth and Katz.

Theorem 6.1 (Guth-Katz). *Let $\mathcal{P} \subset \mathbb{R}^3$ and let $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$ with no B lines of $\mathcal{L}$ in a plane. Then*

$$|I(\mathcal{P}, \mathcal{L})| \lesssim |\mathcal{P}|^{1/2}|\mathcal{L}|^{3/4} + B^{1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{1/3} + |\mathcal{P}| + |\mathcal{L}|.$$

Note that if we take the trivial bound $B = |\mathcal{L}|$ for the maximum number of lines in a plane, then we get back the two-dimensional Szemerédi-Trotter bound, which is tight. The following constructions show that each term in Theorem 6.1 is really necessary. However, this does not quite mean that the bound is completely tight, because we do not have tight constructions for all values of q, in particular not for small q.

Construction 6.2. Let u and v be two parameters, and set

$$\begin{aligned}
\mathcal{P} &:= \{(i, j, k) : i \in [u], j, k \in [2uv]\},\\
\mathcal{L} &:= \{\{(t, at + b, ct + d) : t \in \mathbb{R}\} : a, c \in [v], b, d \in [uv]\}.
\end{aligned}$$

We have $|\mathcal{P}| = 4u^3v^2$ and $|\mathcal{L}| = u^2v^4$. A line $\ell \in \mathcal{L}$ contains at least u points of $\mathcal{P}$. Thus

$$|I(\mathcal{P}, \mathcal{L})| \geq u^3v^4 = (u^3v^2)^{1/2}(u^2v^4)^{3/4} \gtrsim |\mathcal{P}|^{1/2}|\mathcal{L}|^{3/4}.$$

???We still have to show how many lines can lie in a plane, and that for any $|\mathcal{P}|, |\mathcal{L}|$, there exist u and v such that $u^3v^3 \approx |\mathcal{P}|$ and $u^2v^4 \approx |\mathcal{L}|$.

Construction 6.3. Take $|\mathcal{L}|/B$ arbitrary planes, and on each take a configuration of B lines and $|\mathcal{P}|/(|\mathcal{L}|/B)$ points that meets the Szemerédi-Trotter bound. Let $\mathcal{L}$ be the set of all these lines, and $\mathcal{P}$ the set of all these points. Then we have

$$|I(\mathcal{P},\mathcal{L})| \approx \frac{|\mathcal{L}|}{B}\left(\frac{|\mathcal{P}|}{|\mathcal{L}|/B}\right)^{2/3} B^{2/3} = B^{1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{1/3}.$$

We deduce the Guth-Katz Theorem B as stated in the Interlude, via the following more general form.

Corollary 6.4. *Let $k \geq 3$. Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$ with no B in a plane. Then*

$$|\mathcal{I}_{\geq k}(\mathcal{L})| \lesssim \frac{|\mathcal{L}|^{3/2}}{k^2} + B\frac{|\mathcal{L}|}{k^3} + \frac{|\mathcal{L}|}{k}.$$

Proof. Set $\mathcal{P} := \mathcal{I}_{\geq k}(\mathcal{L})$. Then Theorem 6.1 gives

$$k|\mathcal{P}| \lesssim |\mathcal{P}|^{1/2}|\mathcal{L}|^{3/4} + B^{1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{1/3} + |\mathcal{P}| + |\mathcal{L}|.$$

One of the terms on the right must dominate. If it is the first, then $k|\mathcal{P}| \lesssim |\mathcal{P}|^{1/2}|\mathcal{L}|^{3/4}$ gives $|\mathcal{P}| \lesssim k^{-2}|\mathcal{L}|^{3/2}$. Similarly, $k|\mathcal{P}| \lesssim B^{1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{1/3}$ gives $|\mathcal{P}| \lesssim Bk^{-3}|\mathcal{L}|$, and $k|\mathcal{P}| \lesssim |\mathcal{L}|$ gives $|\mathcal{P}| \lesssim k^{-1}|\mathcal{L}|$. If the third term dominates, then we have $k|\mathcal{P}| \lesssim |\mathcal{P}|$, so k is bounded by a constant. In that case, we have to prove the bound $|\mathcal{I}_{\geq k}(\mathcal{L})| \lesssim |\mathcal{L}|^{3/2}$. But this follows from $|\mathcal{I}_{\geq 3}(\mathcal{L})| \lesssim |\mathcal{L}|^{3/2}$, which we proved in Section 3.2 as Theorem 3.10. That theorem was stated over $\mathbb{C}$, but the same statement over $\mathbb{R}$ follows directly. □

Note that to prove this corollary we used Theorem 3.10. We could just as well state this corollary with $k \geq 2$, if instead of Theorem 3.10 we used Theorem 2.3 (and added a condition on the number of lines in a doubly-ruled surface). We have stated it this way to separate it from Theorem A, and in particular to make clear that this theorem does not use the flecnode polynomial or the theory of ruled surfaces.[1]

Corollary 6.5 (Guth-Katz Theorem B)**.** *Let $3 \leq k \leq |\mathcal{L}|^{1/2}$. Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$ with no $|\mathcal{L}|^{1/2}$ in a plane. Then*

$$|\mathcal{I}_{\geq k}(\mathcal{L})| \lesssim \frac{|\mathcal{L}|^{3/2}}{k^2}.$$

6.2 The proof

A key ingredient in the proof of Theorem 6.1 is the polynomial partitioning theorem, proved in Section B.2, which we now state in the form for $\mathbb{R}^3$ that we use.

Theorem 6.6 (Polynomial partitioning in $\mathbb{R}^3$)**.** *Let $\mathcal{P} \subset \mathbb{R}^3$. For any integer $r \geq 1$ there exists a polynomial $F \in \mathbb{R}[x, y, z]$ of degree r such that $\mathbb{R}^3 \backslash Z(F)$ has $O(r^3)$ connected components, each containing $O(|\mathcal{P}|/r^3)$ points of $\mathcal{P}$.*

[1] There is a way to avoid this use of Theorem 3.10. If in Theorem 6.1 we add the condition that all points of $\mathcal{P}$ lie on at least three lines of $\mathcal{L}$, then we could prove the same bound without the $|\mathcal{P}|$ term. Then we could prove Corollary 6.4 without needing Theorem 3.10. I thank Adam Sheffer for pointing this out.

The other key ingredient is the triple point polynomial vector that we already used in Chapter 3. The properties that we need, and their proofs, can be found in Section D.2.

Proof of Theorem 6.1. Let F be a partitioning polynomial of degree r for $\mathcal{P}$, so $\mathbb{R}^3\backslash Z(F)$ has $O(r^3)$ cells, each containing $O(|\mathcal{P}|/r^3)$ points of $\mathcal{P}$. Set $\mathcal{P}_F := \mathcal{P} \cap Z(F)$ and let $\mathcal{L}_F$ be the subset of lines in $\mathcal{L}$ that are contained in $Z(F)$. Points not in $Z(F)$ clearly have no incidences with lines contained in $Z(F)$, so $|I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{L}_F)| = 0$. By Bézout's Inequality (Lemma A.7), a line not contained in $Z(F)$ intersects $Z(F)$ in at most r points, so we have

$$|I(\mathcal{P}_F, \mathcal{L}\backslash\mathcal{L}_F)| \le r|\mathcal{L}|.$$

Next we consider incidences between points not in $Z(F)$ and lines not contained in $Z(F)$. Let $\mathcal{C}$ be the set of all cells of $\mathbb{R}^3\backslash Z(F)$. For each cell $C \in \mathcal{C}$, write $\mathcal{P}_C = \mathcal{P} \cap C$ and $\mathcal{L}_C$ for the set of lines in $\mathcal{L}$ that intersect C. We apply the Szemerédi–Trotter Theorem (Theorem 5.4) inside every cell, which gives

$$|I(\mathcal{P}_C, \mathcal{L}_C)| \lesssim |\mathcal{P}_C|^{2/3}|\mathcal{L}_C|^{2/3} + |\mathcal{P}_C| + |\mathcal{L}_C|.$$

We have $\sum|\mathcal{P}_C| \le |\mathcal{P}|$ and $\sum|\mathcal{L}_C| \le (r+1)|\mathcal{L}|$. We also need the following estimate, which uses Hölder's inequality:

$$\sum_{C\in\mathcal{C}} |\mathcal{P}_C|^{2/3}|\mathcal{L}_C|^{2/3} \le (|\mathcal{P}|/r^3)^{1/3} \cdot \sum |\mathcal{P}_C|^{1/3}|\mathcal{L}_C|^{2/3} \lesssim r^{-1}|\mathcal{P}|^{1/3} \cdot \left(\sum |\mathcal{P}_C|\right)^{1/3} \left(\sum |\mathcal{L}_C|\right)^{2/3}$$
$$\le r^{-1}|\mathcal{P}|^{1/3} \cdot |\mathcal{P}|^{1/3} \cdot (r|\mathcal{L}|)^{2/3} \le r^{-1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3}.$$

Thus we have

$$|I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{L}\backslash\mathcal{L}_F) = \sum_{C\in\mathcal{C}} |I(\mathcal{P}_C, \mathcal{L}_C)| \lesssim r^{-1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + |\mathcal{P}| + r|\mathcal{L}|.$$

Note that, given the terms we have so far, $r = |\mathcal{P}|^{1/2}|\mathcal{L}|^{-1/4}$ would be the right choice of degree for F. We will more or less choose it this way.[2]

It remains to consider the incidences between points on $Z(F)$ and lines contained in $Z(F)$. For this we first decompose $Z(F)$ as

$$Z(F) = Z(G) \cup \bigcup Z(H_i),$$

where $Z(G)$ contains no planes, and each $Z(H_i)$ is a plane. We have $\deg(G) \le \deg(F) = r$ and $\sum \deg(H_i) \le \deg(F) = r$. We will refer to $Z(G)$ as a "component" of $Z(F)$, even though it need not be an irreducible component.

Let us first bound the *external incidences*, by which we mean the incidences $(p, \ell) \in I(\mathcal{P}_F, \mathcal{L}_F)$ where p does not lie on any component of $Z(F)$ that contains ℓ. Given a line

[2] We could also have used the more basic counting that we used in Chapter 5: Each line contains at most $r+1$ incidences that are alone on that line in their cell, and there are at most $r^3 \cdot \binom{|\mathcal{P}|/r^3}{2}$ incidences that are not alone. This gives the bound $r|\mathcal{L}| + |\mathcal{P}|^2/r^3$, which leads to the same choice of r. This mostly works well, but it turns out that this bound is in some sense weaker than the one above, and I was not able to finish the proof as below, in particular in the case $|\mathcal{P}| > |\mathcal{L}|^{3/2}$. This could have been remedied by using Theorem 4.6 to exclude this case, but that would have been unpleasant.

ℓ, the union of the components of $Z(F)$ that do not contain ℓ is a surface of degree at most r, so ℓ intersects this union in at most r points. These intersection points include all external incidences on ℓ. Therefore, the total number of external incidences is bounded by $r|\mathcal{L}|$. It remains to bound only *internal incidences*, i.e., those between a point and a line that both lie on $Z(G)$, or that both lie on a plane $Z(H_i)$.

We have some bookkeeping to set up.

- Set $\mathcal{P}_G := \mathcal{P} \cap Z(G)$, and let $\mathcal{L}_G$ be the set of lines in $\mathcal{L}$ that are contained in $Z(G)$.
- Let $\mathcal{P}_i$ be the subset of points in $\mathcal{P}$ that lie on $Z(H_i)$, but not on any other $Z(H_j)$ or on $Z(G)$; this way we have $\sum |\mathcal{P}_i| \leq |\mathcal{P}|$.
- Let $\mathcal{L}_i$ be the subset of lines in $\mathcal{L}$ that are contained in $Z(H_i)$, but not contained in any other $Z(H_j)$ or in $Z(G)$. Then we have $\sum |\mathcal{L}_i| \leq |\mathcal{L}|$, and by the assumption of the theorem, we have $|\mathcal{L}_i| \leq B$ for each i.
- Let X_1 be the union of all lines in $\mathcal{L}$ that are contained in more than one of the planes $Z(H_i)$, or in a plane $Z(H_i)$ and in $Z(G)$. Let $\mathcal{P}_{X_1} := \mathcal{P} \cap X_1$ and let $\mathcal{L}_{X_1}$ be the set of lines contained in X_1. We have $|\mathcal{L}_{X_1}| \lesssim r^2$. Also note that we actually have $\sum |\mathcal{P}_i| \leq |\mathcal{P} \backslash (\mathcal{P}_G \cup \mathcal{P}_{X_1}|$.

Note that any incidence (p, ℓ) with $p \in X_1$ and $\ell \not\subset X_1$ has already been accounted for as an external incidence, since by definition of X_1, p is contained in some component of $Z(F)$ that does not contain ℓ. So with these definitions we have

$$|I(\mathcal{P}_F, \mathcal{L}_F)| \lesssim r|\mathcal{L}| + |I(\mathcal{P}_G, \mathcal{L}_G)| + \sum |I(\mathcal{P}_i, \mathcal{L}_i)| + |I(\mathcal{P}_{X_1}, \mathcal{L}_{X_1})|.$$

On each plane $Z(H_i)$, we can apply the Szemerédi-Trotter Theorem (Theorem 5.4) to bound the number of internal incidences by

$$|I(\mathcal{P}_i, \mathcal{L}_i)| \lesssim |\mathcal{P}_i|^{2/3}|\mathcal{L}_i|^{2/3} + |\mathcal{P}_i| + |\mathcal{L}_i|.$$

When summing these bounds, we need the following estimate, again using Hölder's inequality:

$$\sum |\mathcal{P}_i|^{2/3}|\mathcal{L}_i|^{2/3} \leq B^{1/3} \cdot \left(\sum |\mathcal{P}_i|\right)^{2/3} \left(\sum |\mathcal{L}_i|\right)^{1/3} \leq B^{1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{1/3}.$$

Hence we have

$$\sum |I(\mathcal{P}_i, \mathcal{L}_i)| \lesssim B^{1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{1/3} + |\mathcal{P}_F \backslash (\mathcal{P}_G \cup \mathcal{P}_{X_1})| + |\mathcal{L}|.$$

It will become apparent later why we need to keep track of the fact that the third term does not involve the points on $Z(G)$ or X_1.

Next we deal with the internal incidences on the plane-free component $Z(G)$. Let T_G be the triple point polynomial vector from Theorem D.1. We have $\deg(\mathsf{T}_G) \lesssim \deg(F) = r$, and we have $\dim(Z(G) \cap Z(\mathsf{T}_G)) \leq 1$ because $Z(G)$ is plane-free; set $X_2 := Z(G) \cap Z(\mathsf{T}_G)$. Set $\mathcal{P}_{X_2} = \mathcal{P}_G \cap X_2$ and let $\mathcal{L}_{X_2}$ be the subset of lines in $\mathcal{L}_G$ that are contained in X_2. By Bézout we have $|\mathcal{L}_{X_2}| \lesssim r^2$.

If $p \notin X_2$, then there are at most two lines on $Z(G)$ that hit p. Thus we have

$$|I(\mathcal{P}_G \backslash \mathcal{P}_{X_2}, \mathcal{L}_G)| \leq 2|\mathcal{P}_G \backslash \mathcal{P}_{X_2}|.$$

As before, a line $\ell \in \mathcal{L}_G$ that is not contained in X_2 intersects X_2 in $O(r)$ points, so we have

$$|I(\mathcal{P}_{X_2}, \mathcal{L}_G \backslash \mathcal{L}_{X_2})| \lesssim r|\mathcal{L}|.$$

Set $X = X_1 \cup X_2$. Also set $\mathcal{P}_X = \mathcal{P} \cap X$ and let $\mathcal{L}_X$ be the subset of lines contained in X. We have $|\mathcal{L}_X| \lesssim r^2$. Altogether, we have

$$|I(\mathcal{P}, \mathcal{L})| \lesssim r^{-1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + r|\mathcal{L}| + B^{1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{1/3} + |\mathcal{P} \backslash \mathcal{P}_X| + |I(\mathcal{P}_X, \mathcal{L}_X)|. \tag{6.1}$$

Our choice of r will depend on the relative sizes of $\mathcal{P}$ and $\mathcal{L}$. The main case is when $|\mathcal{L}|^{1/2} \leq |\mathcal{P}| \leq C_0|\mathcal{L}|^{3/2}$, for a small constant C_0 that is chosen below. In this case we set $r := |\mathcal{P}|^{1/2}|\mathcal{L}|^{-1/4} \geq 1$. Then the first four terms give the desired bound. The last term corresponds to a similar incidence problem, with $|\mathcal{P}_X| \leq |\mathcal{P}|$ and $|\mathcal{L}_X| \lesssim r^2 = |\mathcal{P}||\mathcal{L}|^{-1/2}$. If $|\mathcal{P}|$ is much smaller than $|\mathcal{L}|^{3/2}$, then $|\mathcal{L}_X|$ should be much smaller than $|\mathcal{L}|$. This will allow us to apply induction on $|\mathcal{L}|$, but we have to be careful.

We have proved

$$|I(\mathcal{P}, \mathcal{L})| \leq C_1 \cdot \left(|\mathcal{P}|^{1/2}|\mathcal{L}|^{3/4} + B^{1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{1/3} + |\mathcal{P} \backslash \mathcal{P}_X| + |\mathcal{L}|\right) + |I(\mathcal{P}_X, \mathcal{L}_X)| \tag{6.2}$$

for some constant C_1 that comes from the constants in the proof so far. If we have $|\mathcal{P}| \leq C_0|\mathcal{L}|^{3/2}$ for a sufficiently small C_0, then we have $|\mathcal{L}_X| < |\mathcal{L}|/8$. Then induction gives, for a constant C_2 that we can still choose,

$$\begin{aligned} |I(\mathcal{P}_X, \mathcal{L}_X)| &\leq C_2 \cdot \left(|\mathcal{P}_X|^{1/2}\left(\frac{|\mathcal{L}|}{8}\right)^{3/4} + B^{1/3}|\mathcal{P}_X|^{2/3}\left(\frac{|\mathcal{L}|}{8}\right)^{1/3} + |\mathcal{P}_X| + \left(\frac{|\mathcal{L}|}{8}\right)\right) \\ &\leq \frac{C_2}{2} \cdot \left(|\mathcal{P}|^{1/2}|\mathcal{L}|^{3/4} + B^{1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{1/3} + |\mathcal{L}|\right) + C_2|\mathcal{P}_X|. \end{aligned} \tag{6.3}$$

Choosing $C_2 = 2C_1$ and inserting (6.3) into (6.2) gives

$$\begin{aligned} |I(\mathcal{P}, \mathcal{L})| &\leq 2C_1 \cdot \left(|\mathcal{P}|^{1/2}|\mathcal{L}|^{3/4} + B^{1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{1/3} + |\mathcal{L}|\right) + C_1|\mathcal{P} \backslash \mathcal{P}_X| + C_2|\mathcal{P}_X| \\ &\leq C_2 \cdot \left(|\mathcal{P}|^{1/2}|\mathcal{L}|^{3/4} + B^{1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{1/3} + |\mathcal{P}| + |\mathcal{L}|\right). \end{aligned}$$

This closes the induction. The base case is easy: For either $|\mathcal{P}|$ or $|\mathcal{L}|$ below some constant, $|I(\mathcal{P}, \mathcal{L})| \lesssim |\mathcal{P}| + |\mathcal{L}|$ is trivial.

We have treated the case $|\mathcal{L}|^{1/2} \leq |\mathcal{P}| \leq C_0|\mathcal{L}|^{3/2}$, and now we dispose of the other cases. If $|\mathcal{P}| < |\mathcal{L}|^{1/2}$, then we have $|I(\mathcal{P}, \mathcal{L})| \lesssim |\mathcal{P}|^2 + |\mathcal{L}| \leq |\mathcal{L}|$ by Lemma 5.6, and we are done. If $|\mathcal{P}| > |\mathcal{L}|^2$, then we have $|I(\mathcal{P}, \mathcal{L})| \lesssim |\mathcal{P}| + |\mathcal{L}|^2 \leq |\mathcal{P}|$ by Lemma 5.7, and again we are done.

This leaves us with the case $C_0|\mathcal{L}|^{3/2} \leq |\mathcal{P}| \leq |\mathcal{L}|^2$. We go back to (6.1), but now we set $r = |\mathcal{L}|^{1/2} \geq 1$, which gives

$$r^{-1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + r|\mathcal{L}| = |\mathcal{P}|^{2/3}|\mathcal{L}|^{1/2} + |\mathcal{L}|^{3/2} \lesssim |\mathcal{P}|,$$

so

$$|I(\mathcal{P}, \mathcal{L})| \lesssim B^{1/3}|\mathcal{P}|^{2/3}|\mathcal{L}|^{1/3} + |\mathcal{P} \backslash \mathcal{P}_X| + |\mathcal{L}| + |I(\mathcal{P}_X, \mathcal{L}_X)|.$$

Again we have $|\mathcal{L}_X| \lesssim r^2 \lesssim |\mathcal{L}|$, so we can use induction just as we did using (6.2). This completes the proof. $\square$

Chapter 7

Distinct distances

(Note: This chapter will finish the proof of the Guth-Katz theorem on distinct distances.)

Chapter 8

Point-surface incidences in $\mathbb{R}^3$

In this chapter we prove an incidence theorem for points and surfaces in $\mathbb{R}^3$, and we use it to prove the best known results for unit and distinct distances in $\mathbb{R}^3$. The main new tool that we require to do this is a *second partitioning polynomial*. As we have seen in previous chapters, a partitioning polynomial lets us prove various incidence theorems, but it usually requires some work to deal with incidences that occur on the partitioning hypersurface. In higher dimensions, this only gets harder. The natural way to deal with incidences on a hypersurface is to partition that hypersurface again, with what is called a *second partitioning polynomial*.

8.1 The statement

The main theorem in this chapter is the following. One could think of it as a generalization of the Pach-Sharir Theorem (Theorem 5.8) to surfaces.

Theorem 8.1. *Let $\mathcal{P} \subset \mathbb{R}^3$ and let $\mathcal{S}$ be a set of constant-degree algebraic varieties in $\mathbb{R}^3$. Assume that any s points from $\mathcal{P}$ are contained in at most t varieties from $\mathcal{S}$. Then*

$$|I(\mathcal{P}, \mathcal{S})| \lesssim_{s,t} |\mathcal{P}|^{\frac{2s}{3s-1}} |\mathcal{S}|^{\frac{3s-3}{3s-1}} + |\mathcal{P}| + |\mathcal{S}|.$$

One should think of the varieties in this theorem as surfaces (hence the title of this chapter). Note that in the Appendix we define a surface in $\mathbb{F}^3$ to be any variety of the form $Z(f)$ for $f \in \mathbb{F}[x, y, z]$. Over $\mathbb{R}$ any variety can be written in that way, which is why we have stated the theorem for any varieties. Nevertheless, for one-dimensional varieties, one expects a better bound. For instance, we could apply this theorem to lines, but Theorem 6.1 then gives a significantly better bound.

Let us give a construction that shows that the bound is tight for $s = 3$; for other values of s this is probably not the case (compare with Theorem 5.8, which is thought to be tight only for $s = 2$).

Construction 8.2. Set $\mathcal{P} := [n]^2 \times [n^2] \subset \mathbb{R}^3$ and take the set of paraboloids

$$\mathcal{S} := \left\{ Z((x-i)^2 + (y-j)^2 - (z-k)) : i, j \in [n], k \in [n^2] \right\}.$$

We have $|\mathcal{P}| = |\mathcal{S}| = n^4$. Any three points of $\mathcal{P}$ are contained in at most four surfaces of $\mathcal{S}$. Each surface in $\mathcal{S}$ contains $\Omega(n^2)$ points of $\mathcal{P}$, so we have $|I(\mathcal{P}, \mathcal{S})| \gtrsim n^6$, while Theorem 8.1 gives $|I(\mathcal{P}, \mathcal{S})| \lesssim (n^4)^{3/4}(n^4)^{3/4} = n^6$.

Before embarking on the proof, let us see two basic applications, which should also give us some feeling for the exponents.

Corollary 8.3 (Point-plane incidences). *Let $\mathcal{P} \subset \mathbb{R}^3$ and let $\mathcal{S}$ be a set of planes in $\mathbb{R}^3$.*

- *If no t planes of $\mathcal{S}$ share a line, then*

$$|I(\mathcal{P}, \mathcal{S})| \lesssim |\mathcal{P}|^{4/5}|\mathcal{S}|^{3/5} + |\mathcal{P}| + |\mathcal{S}|;$$

- *If no t points of $\mathcal{P}$ are collinear, then*

$$|I(\mathcal{P}, \mathcal{S})| \lesssim |\mathcal{P}|^{3/5}|\mathcal{S}|^{4/5} + |\mathcal{P}| + |\mathcal{S}|;$$

Corollary 8.4 (Unit distances in $\mathbb{R}^3$). *Let $\mathcal{P} \subset \mathbb{R}^3$. The number of unit distances determined by $\mathcal{P}$ satisfies*

$$|U_3(\mathcal{P})| \lesssim |\mathcal{P}|^{3/2}.$$

8.2 The tools

The main new tool is the following lemma, which we prove in Section B.3.

Lemma 8.5 (Second partitioning polynomial). *Let $Z(F)$ be an irreducible surface in $\mathbb{R}^3$ with $\deg(F) = d$, and let $\mathcal{P}$ be a point set contained in $Z(F)$. Then for every $r \geq d$ there is a polynomial $G \in \mathbb{R}[x, y, z]$, coprime with F, such that $Z(F)\backslash Z(G)$ has $O(dr^2)$ connected components, each containing $O(|\mathcal{P}|/(dr^2))$ points of $\mathcal{P}$.*

We also need refined bounds on the number of connected components of a real variety. More precisely, what we need is bounds on the number of parts that we can get by cutting one variety out of another variety. The following lemmas do exactly that. Recall that for a real variety V, $CC(V)$ denotes the number of connected components of V.

Lemma 8.6. *Let $F, G \in \mathbb{R}[x, y, z]$ with $\deg(F) \leq \deg(G)$. Then*

$$CC(Z(F)\backslash Z(G)) \lesssim \deg(F)^{3-\dim_{\mathbb{R}}(Z(F))} \deg(G)^{\dim_{\mathbb{R}}(Z(F))}.$$

Lemma 8.7. *Let $H_1, H_2, G \in \mathbb{R}[x, y, z]$ with $\deg(H_1) \leq \deg(H_2) \leq \deg(G)$. Then*

$$CC(Z(H_1, H_2)\backslash Z(G)) \lesssim \deg(H_1)^{3-\dim_{\mathbb{R}}(Z(H_1))} \deg(H_2)^{\dim_{\mathbb{R}}(Z(H_1))-\dim_{\mathbb{R}}(Z(H_1,H_2))} \deg(G)^{\dim_{\mathbb{R}}(Z(H_1,H_2))}.$$

8.3 The proof

Proof of Theorem 8.1. Let F be a partitioning polynomial of degree r for $\mathcal{P}$, so $\mathbb{R}^3\backslash Z(F)$ has $O(r^3)$ cells, each containing $O(|\mathcal{P}|/r^3)$ points of $\mathcal{P}$. Set $\mathcal{P}_F := \mathcal{P} \cap Z(F)$.

Let I_1 be the subset of incidences $(p, S) \in I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{S})$ such that S has fewer than s incidences in the cell of p, and let I_2 be the subset of incidences (p, S) such that S has at least s incidences in the cell of p. By Lemma 8.6, a surface in $\mathcal{S}$ intersects $O(r^2)$ cells,???? so we have

$$|I_1| \lesssim_s r^2|\mathcal{S}|.$$

By the usual counting (see the proof of Theorem 5.8), we get

$$|I_2| \lesssim_{s,t} r^3 \cdot \binom{|\mathcal{P}|/r^3}{s} \lesssim_{s,t} \frac{|\mathcal{P}|^s}{r^{3s-3}}.$$

Together these inequalities account for all the incidences in $I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{S})$. At this point, choosing $r = |\mathcal{P}|^{s/(3s-1)}|\mathcal{S}|^{-1/(3s-1)}$ would give the first term in the bound to prove.

We decompose $Z(F)$ as

$$Z(F) = \bigcup Z(F_i),$$

with each F_i irreducible and $\sum \deg(F_i) = r$. Split $\mathcal{P}_F$ into disjoint subsets $\mathcal{P}_i$ such that $\mathcal{P}_i \subset Z(F_i)$.

In Lemma 8.8 below we prove that

$$|I(\mathcal{P}_i, \mathcal{S})| \lesssim_{s,t} \deg(F_i)^{\frac{s-1}{2s-1}}|\mathcal{P}_i|^{\frac{s}{2s-1}}|\mathcal{S}|^{\frac{2s-2}{2s-1}} + |\mathcal{P}_i| + \deg(F_i)^2|\mathcal{S}|. \tag{8.1}$$

Summing up and using Hölder's inequality we get

$$\begin{aligned} |I(\mathcal{P}_F, \mathcal{S})| &\lesssim \left(\sum \deg(F_i)\right)^{\frac{s-1}{2s-1}} \left(\sum |\mathcal{P}_i|\right)^{\frac{s}{2s-1}} |\mathcal{S}|^{\frac{2s-2}{2s-1}} + \sum |\mathcal{P}_i| + \left(\sum \deg(F_i)^2\right) |\mathcal{S}| \\ &\lesssim r^{\frac{s-1}{2s-1}}|\mathcal{P}|^{\frac{s}{2s-1}}|\mathcal{S}|^{\frac{2s-2}{2s-1}} + |\mathcal{P}| + r^2|\mathcal{S}|. \end{aligned}$$

If we choose $r = |\mathcal{P}|^{s/(3s-1)}|\mathcal{S}|^{-1/(3s-1)}$ as predicted above, then the first and third term in this bound coincide, and we get the bound $|\mathcal{P}|^{\frac{2s}{3s-1}}|\mathcal{S}|^{\frac{3s-3}{3s-1}} + |\mathcal{P}|$. We need to ensure that $r \geq 1$, which fails if $|\mathcal{P}|^s < |\mathcal{S}|$. Fortunately, in that case a basic combinatorial argument, exactly like in Lemma 5.6, gives $|I(\mathcal{P}, \mathcal{S})| \lesssim |\mathcal{P}|^s + |\mathcal{S}| \lesssim |\mathcal{S}|$. □

We now prove the bound that we used in (8.1). One can view it as a generalization of the Pach-Sharir Theorem (Theorem 5.8), which is essentially the case $d = 1$, to points and curves on any surface, with the bound worsening with increasing d.

Lemma 8.8. *Let $\mathcal{S}$ be a set of constant-degree algebraic surfaces in $\mathbb{R}^3$. Let $\mathcal{P}$ be a set of points contained in an irreducible surface $Z(F) \subset \mathbb{R}^3$, with $\deg(F) = d$. Assume that any s points from $\mathcal{P}$ are contained in at most t surfaces from $\mathcal{S}$. Then*

$$|I(\mathcal{P}, \mathcal{S})| \lesssim_{s,t} d^{\frac{s-1}{2s-1}}|\mathcal{P}|^{\frac{s}{2s-1}}|\mathcal{S}|^{\frac{2s-2}{2s-1}} + |\mathcal{P}| + d^2|\mathcal{S}|.$$

Proof. The number of surfaces in $\mathcal{S}$ that contain $Z(F)$ is at most t, since any s points in $\mathcal{P}$ would be contained in all these surfaces (assuming that $|\mathcal{P}| \geq s$, but otherwise the bound is trivial). Thus the number of incidences from such surfaces is at most $t|\mathcal{P}| \lesssim |\mathcal{P}|$, and we can assume that no surface in $\mathcal{S}$ contains $Z(F)$.

Apply Lemma 8.5 to get a polynomial G of degree r that is coprime with F, and such that $Z(F)\backslash Z(G)$ has $O(dr^2)$ connected components, each containing $O(|\mathcal{P}|/(dr^2))$ points of $\mathcal{P}$. Set $\mathcal{P}_G := \mathcal{P} \cap Z(G)$.

Since we assumed that no surface $S \in \mathcal{S}$ contains the irreducible surface $Z(F)$, the variety $S \cap Z(F)$ has real dimension less than two. Thus Lemma 8.7 implies that the set $(S \cap Z(F))\backslash Z(G)$ has $O(\deg(S)\deg(F)\deg(G)) = O(dr)$ connected components. This means that each surface in $\mathcal{S}$ intersects $O(dr)$ cells of $Z(F)\backslash Z(G)$. By the usual counting, this gives

$$|I(\mathcal{P}\backslash\mathcal{P}_G, \mathcal{S})| \lesssim dr|\mathcal{S}| + \frac{|\mathcal{P}|^s}{(dr^2)^{s-1}}.$$

We set

$$r := \max\left\{d^{-\frac{s}{2s-1}}|\mathcal{P}|^{\frac{s}{2s-1}}|\mathcal{S}|^{-\frac{1}{2s-1}}, d\right\}.$$

The condition $r \geq d$ is automatically satisfied. The first expression dominates when $d^{3s-1} \leq |\mathcal{P}|^{-s}|\mathcal{S}|$, and in this case we get the first term of the bound to prove. Otherwise we have $r = d$ and we get the third term of the bound to prove.

It remains to deal with the incidences involving points in $Z(F) \cap Z(G)$. In Lemma 8.9 below, we prove that

$$|I(\mathcal{P} \cap Z(F) \cap Z(G), \mathcal{S})| \lesssim_{s,t} |\mathcal{P}| + dr|\mathcal{S}|. \tag{8.2}$$

Although we prove it over $\mathbb{C}$, the statement over $\mathbb{R}$ follows directly. This completes the proof. $\square$

Finally, we prove (8.2) in $\mathbb{C}^3$. We do it over $\mathbb{C}$ because that makes the proof easier; in particular, the notions of dimension and irreducible components behave much better over $\mathbb{C}$ than over $\mathbb{R}$. Another reason is that *we can*, because this bound does not require any polynomial partitioning, and that is the only obstacle that keeps us from doing all these incidence theorems over $\mathbb{C}$. This gives a stronger statement that immediately implies the corresponding statement over $\mathbb{R}$.

Lemma 8.9. *Let $\mathcal{S}$ be a set of constant-degree algebraic surfaces in $\mathbb{C}^3$. Let $\mathcal{P}$ be a set of points contained in a curve $Z(F) \cap Z(G) \subset \mathbb{C}^3$, with F, G coprime and* $\deg(F) = d$, $\deg(G) = e$. *Assume that any s points from $\mathcal{P}$ are contained in at most t surfaces from $\mathcal{S}$. Then*

$$|I(\mathcal{P}, \mathcal{S})| \lesssim_{s,t} |\mathcal{P}| + de|\mathcal{S}|.$$

Proof. The curve

$$C := Z(F) \cap Z(G)$$

has degree at most de, so by Lemma A.6 it has at most de irreducible components. Note that all the components of C have dimension one. Denote the components of C by C_i; we have $\sum \deg(C_i) = \deg(C)$. Partition $\mathcal{P}$ into disjoint sets $\mathcal{P}_i$ so that $\mathcal{P}_i \subset C_i$ (i.e., if a point lies in several C_i, we assign it to one of them arbitrarily). For each $p \in \mathcal{P}$, let $\varphi(p)$ be the number such that $p \in \mathcal{P}_{\varphi(p)}$.

Note that for each i, $C_i \cap S$ is either zero-dimensional or one-dimensional, and in the latter case it must equal C_i, since C_i is irreducible. Let I_1 be the subset of incidences $(p, S) \in I(\mathcal{P}, \mathcal{S})$ such that $C_{\varphi(p)} \cap S$ has dimension zero. Let I_2 be the set of remaining incidences, i.e. the incidences (p, S) such that $C_{\varphi(p)} \subset S$.

First we bound $|I_1|$ using Bézout's Inequality (Theorem A.5). If $(p, S) \in I_1$, then p is a zero-dimensional component of $C_{\varphi(p)} \cap S$. For each $S \in \mathcal{S}$ and each i, the intersection $C_i \cap S$ has degree at most $\deg(C_i) \cdot \deg(S) = O(\deg(C_i))$. Thus $C_i \cap S$ has $O(\deg(C_i))$ irreducible components, and in particular it contains $O(\deg(C_i))$ zero-dimensional components. This gives us

$$|I_1| \lesssim |\mathcal{S}| \cdot \sum \deg(C_i) = |\mathcal{S}| \cdot \deg(C) \leq de|\mathcal{S}|.$$

We can bound $|I_2|$ as follows. Note that if some point $p \in \mathcal{P}_i$ is involved in an incidence $(p, S) \in I_2$, then we have $C_i \subset S$, so in fact $\mathcal{P}_i \subset S$. If $|\mathcal{P}_i| \geq s$, then there are at most t surfaces $S \in \mathcal{S}$ such that $\mathcal{P}_i \subset S$, so the points of $\mathcal{P}_i$ participate in at most $t|\mathcal{P}_i|$ incidences from I_2. On the other hand, if $|\mathcal{P}_i| < s$, then the points of $\mathcal{P}_i$ participate in at most $(s-1)|\mathcal{S}|$ incidences of I_2. Combining these two observations gives

$$|I_2| \leq \sum_i s|\mathcal{S}| + t|\mathcal{P}_i| \lesssim de|\mathcal{S}| + \sum |\mathcal{P}_i| = de|\mathcal{S}| + |\mathcal{P}|.$$

This completes the proof. □

8.4 Distinct distances in $\mathbb{R}^3$

Lemma 8.10. *Let $\mathcal{P} \subset \mathbb{R}^3$ and let $\mathcal{S}$ be a set of constant-degree algebraic surfaces in $\mathbb{R}^3$. Assume that any two points from $\mathcal{P}$ are contained in at most t surfaces from $\mathcal{S}$. Then*

$$|I(\mathcal{P}, \mathcal{S})| \lesssim t^{\frac{2}{5}} |\mathcal{P}|^{\frac{4}{5}} |\mathcal{S}|^{\frac{3}{5}} + t|\mathcal{P}| + |\mathcal{S}|.$$

Proof. We need to trace through the proof of Theorem 8.1 in the case $s = 2$, and keep track of the dependence on t, tedious as it is. The first place we use t is in the bound on $|I_2|$, where we now get $t|\mathcal{P}|^2/r^3$. Equating this with $r^2|\mathcal{S}|$ gives $r = t^{1/5}|\mathcal{P}|^{2/5}|\mathcal{S}|^{-1/5}$, which gives the main term in the bound asserted here. The second place we use t is in the proof of Lemma 8.8, where in the "usual counting" we now get $dr|\mathcal{S}| + t|\mathcal{P}|^2/(dr^2)$. This leads to the choice of $r = t^{1/3}d^{-2/3}|\mathcal{P}|^{2/3}|\mathcal{S}|^{1/3}$, which gives the main term $t^{1/3}d^{1/3}|\mathcal{P}|^{2/3}|\mathcal{S}|^{2/3}$ in the bound of Lemma 8.8. Back in the proof of Theorem 8.1, this changes the main term in the bound on $|I(\mathcal{P}_F, \mathcal{S})|$ to $t^{1/3}r^{1/3}|\mathcal{P}|^{2/3}|\mathcal{S}|^{2/3}$. Plugging in $r = t^{1/5}|\mathcal{P}|^{2/5}|\mathcal{S}|^{-1/5}$ again gives the term that we want. Finally, in Lemma 8.9, we merely have to change the term $|\mathcal{P}|$ to $t|\mathcal{P}|$, and this carries over to the term $t|\mathcal{P}|$ in the bound here. □

Theorem 8.11. *Let $\mathcal{P} \subset \mathbb{R}^3$. Then*

$$|D(\mathcal{P})| \gtrsim \frac{|\mathcal{P}|^{3/5}}{\log |\mathcal{P}|}.$$

Proof. Let Δ be the number of distinct distances determined by $\mathcal{P}$. Then for each point $p \in \mathcal{P}$, we can cover all other points of $\mathcal{P}$ with at most Δ spheres centered at p. Let $\mathcal{S}$ be the set of all these spheres; we have $|\mathcal{S}| \leq \Delta|\mathcal{P}|$ and $|I(\mathcal{P}, \mathcal{S})| = |\mathcal{P}|(|\mathcal{P}| - 1)$.

Let t be the maximum number of points of $\mathcal{P}$ that lie on a plane. From the Guth-Katz theorem we immediately get

$$\Delta \gtrsim \frac{t}{\log t}.$$

On the other hand, t gives an upper bound on the number of spheres of $\mathcal{S}$ through any two points. Indeed, for two points $p, q \in \mathbb{R}^3$, any sphere containing these two points will have its center on a fixed plane (the "bisector plane" of p and q). Thus we can apply Lemma 8.10 to get

$$|I(\mathcal{P}, \mathcal{S})| \lesssim t^{\frac{2}{5}} |\mathcal{P}|^{\frac{4}{5}} (\Delta|\mathcal{P}|)^{\frac{3}{5}} + t|\mathcal{P}| + \Delta|\mathcal{P}|.$$

Hence we have

$$|\mathcal{P}|^2 \lesssim t^{\frac{2}{5}} \Delta^{\frac{3}{5}} |\mathcal{P}|^{\frac{7}{5}},$$

which gives

$$\Delta \gtrsim \frac{|\mathcal{P}|}{t^{\frac{2}{3}}}.$$

It is easy to check that $\max\{|\mathcal{P}|/t^{2/3}, t/\log t\} \geq |\mathcal{P}|^{3/5}/\log|\mathcal{P}|$. Indeed, if one plugs in $t = |\mathcal{P}|^{3/5}$, the smaller of the two is $t/\log t = |\mathcal{P}|^{3/5}/\log|\mathcal{P}|$; since that function is increasing and the other is decreasing, this is a lower bound for the maximum. One can actually do a little better, but this is not that interesting as this bound is not tight anyway. $\square$

Chapter 9

Incidences in $\mathbb{C}^2$

In Section 5.1 we saw a tight bound on incidences between points and lines in $\mathbb{R}^2$. The same statement is true for points and lines in $\mathbb{C}^2$ (it was proved by Tóth [47] and by Zahl [52]), but proving it is considerably more difficult. The reason is that the key tool for our proof in $\mathbb{R}^2$, polynomial partitioning, does not work the same way in $\mathbb{C}^2$: A curve $Z(F) \subset \mathbb{C}^2$ does not partition the space at all, i.e. $\mathbb{C}^2 \backslash Z(F)$ is a connected set. To get around this, we can identify the points and lines in $\mathbb{C}^2$ with points and planes in $\mathbb{R}^4$, where we can use polynomial partitioning. This approach was completed by Zahl [52].

In Section 9.1, we will see a "cheap" version of this proof, which is simpler than Zahl's proof, but gives a slightly worse bound. The proof uses a technique called *constant-degree partitioning*, which was introduced by Solymosi and Tao [39]. Unlike in the proofs so far, where we chose the degree carefully depending on the number of points and varieties, we now choose the degree simply to be a constant. This makes it a bit harder to count incidences inside the cells, but a lot easier to deal with the incidences on the partitioning hypersurface.

9.1 Cheap complex Szemerédi-Trotter

Theorem 9.1. *Let $\mathcal{P} \subset \mathbb{C}^2$ and let $\mathcal{L}$ be a set of lines in $\mathbb{C}^2$. Then*[1]

$$|I(\mathcal{P}, \mathcal{L})| \lesssim_\varepsilon |\mathcal{P}|^{2/3+\varepsilon}|\mathcal{L}|^{2/3} + |\mathcal{P}| + |\mathcal{L}|.$$

The proof is fairly long and involves quite a few calculations, so we first give a more conceptual outline.

Proof sketch. We identify $\mathbb{C}^2$ with $\mathbb{R}^4$, so the lines become planes. We use a partitioning polynomial F of constant degree; this means that the degree does not depend on $|\mathcal{P}|$ or $|\mathcal{L}|$, although it does depend on various constants in the proof, and may still be very large. For the incidences inside the cells of $\mathbb{R}^4 \backslash Z(F)$, we use induction on $|\mathcal{P}|$. We can do this because each cell contains fewer than $|\mathcal{P}|$ points; the drawback is that the induction leads to the extra ε in the exponent of $|\mathcal{P}|$. As usual the challenging part is to handle the incidences on $Z(F)$.

[1] In a theorem statement with a bound involving ε, we implicitly mean that for every $\varepsilon > 0$ there is a choice of constant that makes the statement true.

A plane $\ell \in \mathcal{L}$ either intersects $Z(F)$ in a curve, or it is contained in $Z(F)$. For planes of the first kind, we project the curves to a plane in such a way that the incidence graph is preserved, and then apply the Pach-Sharir incidence theorem. This relies on the fact that F has constant degree; if F had high degree, then so would the intersection curves, which would worsen the Pach-Sharir bound.

For planes contained in $Z(F)$, we use the fact that they come from complex lines, which implies that any two such planes intersect in at most one point. If two such planes are contained in $Z(F)$, their intersection point must be a singular point of $Z(F)$. Thus nonsingular points of $Z(F)$ give only few incidences with planes contained in $Z(F)$. The subset of singular points of $Z(F)$ is a two-dimensional variety X that also has constant degree. A plane that is contained in X is an irreducible component of X, so there is only a constant number of such planes. Finally, for the planes that intersect X in curves, we can again use projection and the Pach-Sharir bound. Note that for each of the steps we heavily rely on the fact that F has constant degree. $\square$

We now begin the proof in full. There are not really big new tools to introduce, other than the idea of constant-degree partitioning. But it is worth pointing out that because we are working in four dimensions, the algebraic geometry aspects become more subtle, and in particular we have to be careful about the distinction between varieties over $\mathbb{R}$ and over $\mathbb{C}$. For this we will frequently refer to Section A.

Proof. We assume $\varepsilon > 0$ is fixed. We use induction on $|\mathcal{P}|$ with the induction hypothesis

$$|I(\mathcal{P},\mathcal{L})| \le C_1|\mathcal{P}|^{2/3+\varepsilon}|\mathcal{L}|^{2/3} + C_2|\mathcal{P}| + C_2|\mathcal{L}|.$$

For the induction base, we observe that the bound holds for $|\mathcal{P}|$ sufficiently small and C_1, C_2 sufficiently large.[2]

Let $F \in \mathbb{R}[w,x,y,z]$ be a partitioning polynomial of degree r as in Theorem B.6, so $\mathbb{R}^4\backslash Z(F)$ has C_3r^4 connected components, each containing $C_4|\mathcal{P}|/r^4$ points of $\mathcal{P}$, for constants C_3, C_4. Set $\mathcal{P}_F := \mathcal{P} \cap Z(F)$ and let $\mathcal{L}_F$ be the subset of planes in $\mathcal{L}$ that are contained in $Z(F)$.

Let $\mathcal{O}$ be the set of connected components of $\mathbb{R}^4\backslash Z(F)$, and for each cell $O \in \mathcal{O}$, set $\mathcal{P}_O := \mathcal{P} \cap O$ and let $\mathcal{L}_O$ be the set of planes that intersect O. In each cell O, we apply the induction hypothesis to get

$$|I(\mathcal{P}_O,\mathcal{L}_O)| \le C_1|\mathcal{P}_O|^{2/3+\varepsilon}|\mathcal{L}_O|^{2/3} + C_2|\mathcal{P}_O| + C_2|\mathcal{L}_O|.$$

Summing over all C_3r^4 cells gives

$$\begin{aligned}|I(\mathcal{P}\backslash\mathcal{P}_F,\mathcal{L}\backslash\mathcal{L}_F)| &\le C_1\sum_{O\in\mathcal{O}}|\mathcal{P}_O|^{2/3+\varepsilon}|\mathcal{L}_O|^{2/3} + C_2\sum_{O\in\mathcal{O}}|\mathcal{P}_O| + C_2\sum_{O\in\mathcal{O}}|\mathcal{L}_O|\\ &\le C_1\left(C_4\frac{|\mathcal{P}|}{r^4}\right)^{2/3+\varepsilon}\sum_{O\in\mathcal{O}}|\mathcal{L}_O|^{2/3} + C_2|\mathcal{P}\backslash\mathcal{P}_F| + C_2\sum_{O\in\mathcal{O}}|\mathcal{L}_O|.\end{aligned}$$

[2] Because this proof involves some juggling with constants, we avoid asymptotic notation and keep track of the various constants.

A simple application of Theorem A.12 shows that each plane intersects $C_5 r^2$ of the $C_3 r^4$ cells, for some constant C_5.[3] This means that $\sum |\mathcal{L}_O| \le C_5 r^2 |\mathcal{L}|$ and, using Hölder's inequality,

$$\sum_{O\in\mathcal{O}} |\mathcal{L}_O|^{2/3} \le \left(\sum_{O\in\mathcal{O}} |\mathcal{L}_O|\right)^{2/3} \left(\sum_{O\in\mathcal{O}} 1\right)^{1/3} \le (C_5 r^2|\mathcal{L}|)^{2/3}\cdot (C_3 r^4)^{1/3} = C_6 r^{8/3}|\mathcal{L}|^{2/3}.$$

Thus we get

$$\begin{aligned}|I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{L}\backslash\mathcal{L}_F)| &\le C_1\left(C_4\frac{|\mathcal{P}|}{r^4}\right)^{2/3+\varepsilon}\cdot C_6 r^{8/3}|\mathcal{L}|^{2/3} + C_2|\mathcal{P}\backslash\mathcal{P}_F| + C_2C_5r^2|\mathcal{L}|\\ &\le C_1C_4^{2/3+\varepsilon}C_5 r^{-4\varepsilon}|\mathcal{P}|^{2/3+\varepsilon}|\mathcal{L}|^{2/3} + C_2|\mathcal{P}\backslash\mathcal{P}_F| + C_2C_5r^2|\mathcal{L}|.\end{aligned}$$

Since C_4, C_5, ε are fixed, we can choose r large enough so that $C_4^{2/3+\varepsilon}C_5 r^{-4\varepsilon} \le 1/4$. To deal with the last term, we force $C_2C_5r^2|\mathcal{L}| \le C_1|\mathcal{P}|^{2/3+\varepsilon}|\mathcal{L}|^{2/3}/4$, which requires that we have $(4C_1^{-1}C_2C_5r^2)^3|\mathcal{L}| \le |\mathcal{P}|^{2+3\varepsilon}$. We can assume that $|\mathcal{L}| \le |\mathcal{P}|^2$, since otherwise we are done by Lemma E.1. Then we choose C_1 large enough so that $4C_1^{-1}C_2C_5r^2 \le 1$. This gives us

$$|I(\mathcal{P}\backslash\mathcal{P}_F, \mathcal{L}\backslash\mathcal{L}_F)| \le \frac{C_1}{2}|\mathcal{P}|^{2/3+\varepsilon}|\mathcal{L}|^{2/3} + C_2|\mathcal{P}\backslash\mathcal{P}_F|. \tag{9.1}$$

It remains to deal with the incidences in $I(\mathcal{P}_F, \mathcal{L}\backslash\mathcal{L}_F)$ and $I(\mathcal{P}_F, \mathcal{L}_F)$.

Define

$$\mathcal{C} := \{\ell \cap Z(F) : \ell \in \mathcal{L}\backslash\mathcal{L}_F\}.$$

Since a plane $\ell \notin \mathcal{L}_F$ is irreducible and not contained in $Z(F)$, it follows from Lemma A.2 that $\dim(\ell\cap Z(F)) < \dim(\ell) = 2$. Hence the sets in $\mathcal{C}$ are curves (or perhaps finite sets), and they still have the property that any two points are contained in at most one curve. We will prove in Theorem 9.2 below that this essentially allows us to apply the Pach-Sharir bound (see Theorem 5.8), which gives

$$|I(\mathcal{P}_F, \mathcal{C})| \lesssim |\mathcal{P}_F|^{2/3}|\mathcal{C}|^{2/3} + |\mathcal{P}_F| + |\mathcal{C}|,$$

or in other words

$$|I(\mathcal{P}_F, \mathcal{L}\backslash\mathcal{L}_F)| \le C_7|\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + C_7|\mathcal{P}_F| + C_7|\mathcal{L}|. \tag{9.2}$$

We have kept the second term as $|\mathcal{P}_F|$ for a reason that will become clear below.

Now for $I(\mathcal{P}_F, \mathcal{L}_F)$. Consider a nonsingular point $p \in Z(F)$, and suppose that it is contained in two planes $\ell, \ell' \subset Z(F)$. Both planes would have to be contained in the tangent space of $Z(F)$ at p, which is a 3-flat. But this is impossible: Since ℓ_1, ℓ_2 come from complex lines, they intersect in a single point (namely p), which means that as planes they cannot be contained in the same 3-flat. Therefore, a nonsingular point $p \in \mathcal{P}_F$ can

[3] In fact, we do not need Theorem A.12 here; we can view a plane ℓ as a copy of $\mathbb{R}^2$, and then use the simpler Theorem A.10 to bound the number of connected components of $\ell\backslash Z(F)$, which also gives $O(r^2)$ components.

be involved in at most one incidence from $I(\mathcal{P}_F, \mathcal{L}_F)$. If we write $\mathcal{P}_{\text{sing}}$ for the points in $\mathcal{P}$ that are singular points of $Z(F)$, then we have

$$|I(\mathcal{P}_F \backslash \mathcal{P}_{\text{sing}}, \mathcal{L}_F)| \leq |\mathcal{P}_F|. \tag{9.3}$$

Finally, we have to deal with $I(\mathcal{P}_{\text{sing}}, \mathcal{L}_F)$. We will use Lemma A.3, which means we have to argue over $\mathbb{C}$. Let $X \subset Z_{\mathbb{R}^4}(F)$ be the subset of singular points of $Z_{\mathbb{R}^4}(F)$, and let $X_{\mathbb{C}}$ be the subset of singular points of $Z_{\mathbb{C}^4}(F)$; we have $X \subset X_{\mathbb{C}}$. By Lemma A.3, we have $\dim_{\mathbb{C}}(X_{\mathbb{C}}) < \dim_{\mathbb{C}}(Z_{\mathbb{C}^4}(F)) = 3$, so $\dim_{\mathbb{C}}(X_{\mathbb{C}}) \leq 2$, which implies $\dim_{\mathbb{R}}(X) \leq 2$. Lemma A.1 also states that $X_{\mathbb{C}}$ has degree $O_r(1)$, so it has $O_r(1)$ irreducible components, and in particular it contains $O_r(1)$ complex planes. This implies that X contains $O_r(1)$ planes from $\mathcal{L}$. Together these give $C_8|\mathcal{P}_F|$ incidences for some constant C_8 depending on r. The other planes intersect X in curves or finite sets. Again applying Theorem 9.2 below, we get

$$|I(\mathcal{P}_{\text{sing}}, \mathcal{L}_F)| \leq C_8|\mathcal{P}_F| + C_7|\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + C_7|\mathcal{P}_F| + C_7|\mathcal{L}|. \tag{9.4}$$

Combining (9.1), (9.2), (9.3), and (9.4) we have

$$|I(\mathcal{P}, \mathcal{L})| \leq \frac{C_1}{2}|\mathcal{P}|^{2/3+\varepsilon}|\mathcal{L}|^{2/3} + C_2|\mathcal{P}\backslash\mathcal{P}_F| + 2C_7|\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + (2C_7 + C_8 + 1)|\mathcal{P}_F| + 2C_7|\mathcal{L}|.$$

We can choose C_2 sufficiently large so that $2C_7 + C_8 + 1 \leq C_2$, and then we can choose C_1 large enough so that $2C_7 \leq C_1/2$ (along with the earlier restriction on C_1, namely that $C_1 \geq 4C_2C_5r^2$). This results in

$$|I(\mathcal{P}, \mathcal{L})| \leq C_1|\mathcal{P}|^{2/3+\varepsilon}|\mathcal{L}|^{2/3} + C_2|\mathcal{P}| + C_2|\mathcal{L}|,$$

completing the induction step and the proof. □

Theorem 9.2. *Let $\mathcal{P} \subset \mathbb{R}^D$ and let $\mathcal{C}$ be a set of constant-degree algebraic curves in $\mathbb{R}^D$. Suppose that any s points of $\mathcal{P}$ are contained in at most t curves of $\mathcal{C}$. Then*

$$|I(\mathcal{P}, \mathcal{C})| \lesssim_{s,t} |\mathcal{P}|^{\frac{s}{2s-1}}|\mathcal{C}|^{\frac{2s-2}{2s-1}} + |\mathcal{P}| + |\mathcal{C}|.$$

Proof. By Lemma A.13, there exists a projection $\pi : \mathbb{R}^D \to \mathbb{R}^2$ such that if we set

$$\mathcal{P}' := \pi(\mathcal{P}), \quad \mathcal{C}' := \left\{\overline{\pi(C)} : C \in \mathcal{C}\right\},$$

then $|\mathcal{P}'| = |\mathcal{P}|$, $|\mathcal{C}'| = |\mathcal{C}|$, and most importantly, the incidence graph $I(\mathcal{P}', \mathcal{C}')$ is isomorphic to $I(\mathcal{P}, \mathcal{C})$. In particular, any s points of $\mathcal{P}'$ are contained in at most t curves of $\mathcal{C}'$. Moreover, the curves in $\mathcal{C}'$ also have constant degree. It follows that we can use Theorem 5.8 to get the claimed bound. □

Appendix A

Basic algebraic geometry

In this chapter we introduce the tools from algebraic geometry that we use in the main text. There are very few proofs here, but we try to provide references, usually to the textbooks *Algebraic Geometry* by Harris [19] or *Ideals, Varieties, and Algorithms* by Cox, Little, and O'Shea [6]. There are many other textbooks, but [19] includes most of the results that are relevant to us, without too much technical machinery; [6] is somewhat easier, and also includes computational techniques.

The only fields we will consider are $\mathbb{C}$ and $\mathbb{R}$. Since combinatorial geometry questions most often involve $\mathbb{R}$, we will emphasize the distinctions between $\mathbb{C}$ and $\mathbb{R}$, and how they can be overcome. A few of the facts that we need can easily be proved over an arbitrary field, and then we write $\mathbb{F}$ to denote an arbitrary field. Because algebraic geometry works best over algebraically closed fields, most algebraic geometry texts focus on algebraically closed fields; this is the case for [19], while [6] does consider other fields when feasible.

A.1 Definitions and basic facts

In this section we define some basic concepts from algebraic geometry and list their basic properties. If possible, we introduce concepts over an arbitrary field $\mathbb{F}$, but in many cases this can be problematic, and then we restrict to $\mathbb{C}$, which is usually the ideal setting. In Section A.3, we will study the corresponding notions over $\mathbb{R}$, which often require more care.

Varieties. A **variety** in $\mathbb{F}^D$ is a set of the form

$$Z_{\mathbb{F}^D}(F_1, \ldots, F_k) := \left\{(x_1, \ldots, x_D) \in \mathbb{F}^D : F_i(x_1, \ldots, x_D) = 0 \text{ for } i = 1, \ldots, k\right\},$$

for polynomials $F_i \in \mathbb{F}[x_1, \ldots, x_D]$. Such sets are normally called *affine* varieties, to distinguish them from *projective* varieties, but since we will mostly focus on affine varieties, we refer to them simply as varieties. If X and Y are varieties, then $X \cup Y$, $X \cap Y$ and $X \times Y$ are also varieties. A **subvariety** of a variety X is a subset of X that is a variety. A subvariety of X is **proper** if it is neither X nor the empty set. A variety is **irreducible** if it is not the union of two proper subvarieties. The following property is very useful (a proof can be found in Cox-Little-O'Shea [6, Theorem 4.6.4]).

Lemma A.1. *Every variety has a decomposition into finitely many irreducible components. More precisely, given a variety $X \subset \mathbb{F}^D$, we can write $X = X_1 \cup \cdots \cup X_k$, with each X_i an irreducible variety, and no X_i contained in another X_j. This decomposition is unique (except for the order of the components).*

The simplest type of variety is a **hypersurface**, which is a variety that can be written in the form $Z(F)$ for a nonconstant F.[1] The **degree** of a hypersurface H is the degree of a minimum-degree polynomial F such that $H = Z(F)$. A hypersurface in $\mathbb{F}^2$ is usually called a **curve**, and a hypersurface in $\mathbb{F}^3$ is called a **surface**. If F is linear, then the hypersurface $Z(F)$ is an affine space of (linear) dimension $D-1$. More generally, we refer to an affine subspace of dimension k as a k-**flat**.

Note that one has to be careful with these terms over fields that are not algebraically closed. For instance, the variety $Z_{\mathbb{R}^2}(x^2+y^2)$ is technically a hypersurface, but it consists of a single point, so probably shouldn't be called a curve. In combinatorial questions this often does not matter, but when there is risk of confusion one should make the definition clear.

Dimension. The **dimension** $\dim_{\mathbb{F}}(X)$ of a variety X can be defined as the largest D such that there is a proper sequence $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_D \subset X$ of irreducible varieties X_i. This definition is easy to state but pretty useless in practice. There are many equivalent definitions, see for instance [19, Lecture 11] for several definitions (over an algebraically closed field). Usually we do not need the definition, but only some basic properties of dimension. We first state these over $\mathbb{C}$, where dimension behaves well.

- For varieties X, Y, we have $\dim_{\mathbb{C}}(X \cup Y) = \max\{\dim_{\mathbb{C}}(X), \dim_{\mathbb{C}}(Y)\}$.
- If $F \in \mathbb{C}[x_1, \ldots, x_D] \backslash \mathbb{C}$, then $\dim_{\mathbb{C}}(Z(F)) = D-1$.
- If $F, G \in \mathbb{C}[x_1, \ldots, x_D] \backslash \mathbb{C}$ are coprime, then $\dim_{\mathbb{C}}(Z(F,G)) = D-2$ or $Z(F,G)$ is empty.
- Let $X \subset \mathbb{C}^D$ be an irreducible variety and $F \in \mathbb{C}[x_1, \ldots, x_D]$. Then
$$\dim_{\mathbb{C}}(X \cap Z(F)) = \dim_{\mathbb{C}}(X) - 1,$$
unless $X \cap Z(F) = \emptyset$ or $X \cap Z(F) = X$.
- If a nonempty variety $X \subset \mathbb{C}^D$ (not necessarily irreducible) is defined by ℓ polynomials, then $\dim_{\mathbb{C}}(X) \geq D - \ell$.

Over an arbitrary field, and in particular over $\mathbb{R}$, most of these properties can fail. One fact that does hold reliably is the following.

Lemma A.2. *Let $X, Y \subset \mathbb{F}^D$ be varieties with X irreducible and $X \not\subset Y$. Then*
$$\dim_{\mathbb{F}}(X \cap Y) < \dim_{\mathbb{F}}(X).$$

[1] Occasionally, one has to be careful that the polynomial defining a hypersurface H is of minimum degree, i.e. F is the minimum-degree polynomial such that $H = Z(F)$. For instance, $Z(F) = Z(F^2)$, but some typical properties may fail with the polynomial F^2. Therefore, when discussing a hypersurface $Z(F)$, we will assume F to be minimum-degree without further comment. In particular, this implies that F is *squarefree*, i.e., none of its factors is a square of a polynomial.

Degree. We have already defined the degree of a hypersurface over any field. For varieties of dimension lower than $D-1$, the degree is less easy to define, and we only do it over $\mathbb{C}$. We define the **degree** $\deg(X)$ of a k-dimensional irreducible variety X in $\mathbb{C}^D$ by

$$\deg(X) := \max\{|X \cap L| : L \text{ is a } (D-k)\text{-flat such that } X \cap L \text{ is finite}\}.$$

For a reducible variety X, we define $\deg(X)$ to be the sum of the degrees of its irreducible components (even if these components have different dimensions); it thus directly follows that the number of irreducible components of X is at most $\deg(X)$. If X is a flat we have $\deg(X) = 1$, for a finite set X we have $\deg(X) = |X|$, and for a hypersurface we have $\deg(Z(f)) = \deg(f)$. Another basic property that we often use is that if X is irreducible, $X \subset Y$, and $\dim(X) = \dim(Y)$, then $\deg(X) \leq \deg(Y)$. More generally, if $X \subset Y$ and every irreducible component of X has the same dimension as the irreducible component of Y that it is contained in, then $\deg(X) \leq \deg(Y)$.

Singularities. For a hypersurface $Z(f)$ in $\mathbb{F}^D$, a **singularity** is a point $p \in Z(f)$ such that $(\nabla f)(p) = 0$, where ∇f is the gradient $\nabla f := (\partial f/\partial x_1, \ldots, \partial f/\partial x_D)$.[2] It is a point where the hypersurface does not have a tangent hyperplane (although we do not define that here). For varieties that are not hypersurfaces, singularities are harder to define (see for instance [19, Lecture 14] for an algebraic definition). We will cheat a bit and refer to differential geometry: A **singularity** of a variety over $\mathbb{C}$ is a point that does not have a neighborhood in which the variety coincides with a smooth complex manifold (see [19, Exercise 14.1]). The following lemma gives a key property (see [19, Exercise 14.3] or [6, Theorem 9.6.8]).

Lemma A.3. *The set X_{sing} of singular points of a variety $X \subset \mathbb{C}^D$ is a subvariety of X with $\dim(X_{sing}) < \dim(X)$ and $\deg(X_{sing}) \lesssim_{\deg(X)} 1$.*

A.2 Degree bounds

In combinatorial applications, we often need rough bounds on the "complexity" of a variety, which roughly means its dimension, degree, the number of polynomials defining the variety and their degrees, or combinations thereof. The fact that such bounds exist is in fact what makes these applications possible.

The prototype bound is the following theorem. Recall that the degree of a finite point set is its cardinality, so the lemma says that the intersection of two curves is either a finite set of bounded size, or contains a curve of bounded degree (a common component). A proof of this theorem over an arbitrary field can be found in Gibson [14, Lemma 14.4]. Recall that for curves in $\mathbb{F}^2$ we defined the degree to be the degree of a minimum-degree defining polynomial.

Theorem A.4 (Bézout's Inequality). *If C_1 and C_2 are algebraic curves in $\mathbb{F}^2$, then*

$$\deg(C_1 \cap C_2) \leq \deg(C_1) \cdot \deg(C_2).$$

[2] Note that here it is important that F is squarefree, since if F had a square factor, then F would have a common factor with its derivatives, and thus every point of $Z(F)$ would be singular.

Over $\mathbb{C}$, this bound generalizes nicely. A proof of the projective version can be found in most algebraic geometry textbooks, including [19, Lecture 18]. We focus on affine varieties, and a version for affine varieties can be found in Heintz [21, Theorem 1] or Schmid [33]. Other convenient forms with short proofs are in Tao [48].

Theorem A.5 (Bézout's Inequality in $\mathbb{C}^D$)**.** *If X and Y are varieties in $\mathbb{C}^D$, then*

$$\deg(X \cap Y) \leq \deg(X) \cdot \deg(Y).$$

The following bound is an immediate consequence of Theorem A.5. Since the degree of a reducible variety is the sum of the degrees of its irreducible components, this lemma also gives a bound on the number of irreducible components of a variety.

Lemma A.6. *If $X = Z_{\mathbb{C}^D}(f_1, \ldots, f_k)$, then*

$$\deg(X) \leq \prod_{i=1}^{k} \deg(f_i).$$

The right hand side also bounds the number of irreducible components of X.

In these notes we often use the following corollary. It is in fact easy to prove over any field without using Theorem A.5, and we do that here.

Lemma A.7. *If L is a line in $\mathbb{F}^D$ and $Z(F) \subset \mathbb{F}^D$, then $|Z(F) \cap L| \leq \deg(F)$ or $L \subset Z(F)$.*

Proof. We can parametrize $L = \{p + tv : t \in \mathbb{F}\}$ for a point p and a direction v (this is just linear algebra, so works over any field). Then the points in $Z(F) \cap L$ correspond to roots of the univariate polynomial $F(p + tv)$, which has degree at most $\deg(F)$ in t. Either there are at most $\deg(F)$ such roots, or $F(p + tv)$ is identically zero, so $L \subset Z(F)$. $\square$

Another corollary that we often use is the following; again we prove it over an arbitrary field, just because we can.

Lemma A.8. *Let $F, G \in \mathbb{F}[x, y, z]$ be two coprime polynomials. The number of lines contained in $Z(F) \cap Z(G)$ is at most* $\deg(F) \cdot \deg(G)$.

Proof. By Lemma A.1, the variety $Z(F) \cap Z(G)$ consists of a finite number of irreducible components. Lemma A.2 implies that each of these components has dimension at most one, since a component of dimension two would correspond to a common factor of F and G. Choose a plane $\pi \subset \mathbb{F}^3$ such that it does not contain any of these components, but it does intersect each component that is a line. This implies that $Z(F) \cap Z(G) \cap \pi$ is finite, and that its cardinality is an upper bound on the number of lines in $Z(F) \cap Z(G)$. The varieties $Z(F) \cap \pi$ and $Z(G) \cap \pi$ are plane curves of degree at most $\deg(F)$ and $\deg(G)$, since we can obtain equations for them by plugging in the equation for π. Thus we can apply Theorem A.4 to these curves to get $|Z(F) \cap Z(G) \cap \pi| \leq \deg(F) \cdot \deg(G)$. $\square$

A.3 Real algebraic geometry

We now discuss real analogues of some of the statements that we saw above only over $\mathbb{C}$. One important fact to keep in mind is that over $\mathbb{R}$, *every variety is a hypersurface*. Indeed, we have

$$Z_{\mathbb{R}^D}(f_1, \dots, f_k) = Z_{\mathbb{R}^D}(f_1^2 + \cdots + f_k^2).$$

This makes a number of facts that we saw over $\mathbb{C}$ suspicious over $\mathbb{R}$. For instance, it cannot be true that every hypersurface in $\mathbb{R}^D$ has dimension $D-1$.

Nevertheless, we can still obtain a bound on the number of irreducible components of a variety, comparable to Lemma A.6.

Lemma A.9. *Let $X \subset \mathbb{R}^D$ be a variety defined by polynomials of degree at most d. Then X consists of $O_d(1)$ irreducible components.*

Dimension. For real varieties, the dimension is a trickier concept than for complex varieties. We have given an abstract definition above, but we refer to [3, Section 5.3] for a careful definition of the real dimension $\dim_{\mathbb{R}}(X)$ of a real variety X. Roughly, we can define the dimension of X to be the maximum D such that there exists an injective homeomorphism $[0,1]^D \to X$. Alternatively, we can resort to differential geometry: One can view a real variety as a smooth real manifold (away from singularities), and the real dimension then equals the dimension in the manifold sense.

Complexity bounds. In $\mathbb{R}^D$ for $D > 2$, Bézout's Inequality as stated above can fail. Take for instance

$$F_1 := (x(x-1)(x-2))^2 + (y(y-1)(y-2))^2, \quad F_2 := z.$$

Then $\deg(Z_{\mathbb{R}^3}(F_1, F_2)) = 9$, but $\deg(F_1) \cdot \deg(F_2) = 6$.

Nevertheless, in practice we can usually achieve similar results with a topological bound. For instance, the following theorem bounds the number of connected components of a real variety and of its complement in terms of the degrees of its defining polynomials. The first part is due to Oleinik and Petrovskii [27], Milnor [26], and Thom [49], and the second part is due to Warren [50]. We write $CC(X)$ for the number of connected components of a set X.

Theorem A.10. *Let $X = Z_{\mathbb{R}^D}(f_1, \dots, f_k)$ and assume that $\deg(f_i) \le d$ for each i. Then*

$$CC(X) \le (2d)^D \quad \text{and} \quad CC(\mathbb{R}^D \backslash X) \le (2d)^D.$$

This bound depends on a single upper bound for the degrees of the polynomials. Bézout's Inequality would lead us to believe that such bounds should take into account the individual degrees of the polynomials, but this is not easy to make precise. It was done in 2013 by Barone and Basu [1]. We simplify the statement of their result somewhat using the following (non-standard) definition.

Definition A.11. Consider $X = Z_{\mathbb{R}^D}(g_1, \dots, g_m)$ with $\deg(g_1) \le \cdots \le \deg(g_m)$. Write $\dim_{\mathbb{R}}(Z_{\mathbb{R}^D}(g_1, \dots, g_i)) = k_i$ and $k_0 = D$. We define the *Barone-Basu degree* of X by

$$\deg_{\mathrm{BB}}(X) = \prod_{i=1}^{m} \deg(g_i)^{k_{i-1} - k_i}.$$

Theorem A.12 (Barone-Basu). *Let $X = Z_{\mathbb{R}^D}(g_1, \dots, g_m)$ with $\deg(g_1) \le \cdots \le \deg(g_m)$. Let $h \in \mathbb{R}[x_1, \dots, x_D]$ with $\deg(h) \ge \deg(g_m)$. Then the number of connected components of both $X \cap Z_{\mathbb{R}^D}(h)$ and $X \backslash Z_{\mathbb{R}^D}(h)$ is*

$$O\left(\deg_{BB}(X) \cdot \deg(h)^{\dim_{\mathbb{R}}(X)}\right).$$

In the ideal case where each $k_i = D - i$, this would be a natural generalization of Theorem A.5, or more specifically Lemma A.6. On the other hand, if $\deg(g_i) \le d$ for each i, we get the bound $O(d^{D-k_m} \deg(h)^{k_m})$, without any individual conditions on the g_i.

Singularities. The study of singularities is also more subtle over $\mathbb{R}$. Consider the curve $Z(x^3 + x^2y + x^4)$. Its gradient vanishes at the origin, but if you plot the curve, it will look wonderfully smooth there.[3] Thus, unlike over $\mathbb{C}$, the vanishing of the gradient does not correspond exactly to the real variety not being a smooth manifold around the point. Nevertheless, we will stick with the definition of a singularity on a hypersurface that we have given above.

Note that many hypersurfaces can be entirely singular. Take coprime $F, G \in \mathbb{R}[x, y, z]$. Then $Z_{\mathbb{R}^3}(F^2 + G^2)$ has a gradient that vanishes on all points of the variety.

A.4 Projections

Lemma A.13. *Let $\mathcal{P} \subset \mathbb{R}^D$ and let $\mathcal{C}$ be a set of constant-degree algebraic curves in $\mathbb{R}^D$. There exists a projection $\pi : \mathbb{R}^D \to \mathbb{R}^2$ such that if we set*

$$\mathcal{P}' := \pi(\mathcal{P}), \quad \mathcal{C}' := \left\{\overline{\pi(C)} : C \in \mathcal{C}\right\},$$

then the following hold.
(i) $|\mathcal{P}'| = |\mathcal{P}|$ and $|\mathcal{C}'| = |\mathcal{C}|$;
(ii) The curves in $\mathcal{C}'$ have constant degree;
(iii) The incidence graph $I(\mathcal{P}', \mathcal{C}')$ is isomorphic to $I(\mathcal{P}, \mathcal{C})$.

(Note: Proof to be added.)

[3] The reason is that in $\mathbb{C}^2$ the curve intersects itself at the origin, but one of the branches is complex and does not show up in the real plane. The gradient does not care and considers this a singularity.

Appendix B

Interpolation and partitioning

We will use the Veronese map $V = V_{D,d} : \mathbb{F}^D \to \mathbb{F}^{\binom{D+d}{D}}$, defined by

$$V_{D,d}(x_1, \dots, x_D) = (x_1^{i_1} \cdots x_D^{i_D})_{i_1+\dots+i_D \le d}.$$

The map is injective and its image is a D-dimensional subvariety of $\mathbb{F}^{\binom{D+d}{D}}$. Let us write z_I for the coordinates of $\mathbb{F}^{\binom{D+d}{D}}$, for any $I \in \{(i_1, \dots, i_D) : i_1 + \dots + i_D \le d\}$, and x^I for the monomial $x_1^{i_1} \cdots x_D^{i_D}$. Note that $z_{(0,\cdots,0)} = 1$ and $x^{(0,\cdots,0)} = 1$. The key observation is that a hyperplane $\sum a_I z_I = 0$ in $\mathbb{F}^{\binom{D+d}{D}}$ corresponds to a hypersurface $\sum a_I x^I = 0$ in $\mathbb{F}^D$, in the sense that $x \in \mathbb{F}^D$ lies on the hypersurface if and only if $V_{D,d}(x)$ lies on the hyperplane.

For instance, the map defined by

$$V_{2,2}(x, y) = (x, y, x^2, xy, y^2)$$

maps $\mathbb{F}^2$ to a surface in $\mathbb{F}^5$, and the points on a conic $a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} = 0$ are mapped to the points on the hyperplane $a_{20}z_{20} + a_{11}z_{11} + a_{02}z_{02} + a_{10}z_{10} + a_{01}z_{01} + a_{00} = 0$.

B.1 Interpolation

Lemma B.1. *Let $S \subset \mathbb{F}^D$ be a finite set. Then there exists a squarefree $F \in \mathbb{F}[x_1, \dots, x_D] \backslash \mathbb{F}$ such that $S \subset Z_{\mathbb{F}^D}(F)$ and $\deg(F) \le D|S|^{1/D}$.*

Proof. Let d be an integer such that $\binom{D+d}{D} > |S|$. Then the finite set

$$V_{D,d}(S) \subset \mathbb{F}^{\binom{D+d}{d}}$$

has cardinality smaller than the dimension of the vector space it is in. That implies there is an affine hyperplane $\sum a_I z_I = 0$ containing $V_{D,d}(S)$. For a point $(x^I)_I \in V_{D,d}(S)$, this means that $\sum a_I x^I = 0$. Therefore, if we set $F(x_1, \dots, x_D) = \sum a_I x^I$, then $S \subset Z_{\mathbb{F}^D}(F)$.

If we choose $d = \lfloor D|S|^{1/D} \rfloor$, then $d^D > D!|S|$, so $|S| < d^D/D! < \binom{D+d}{D}$. If F is not squarefree, we can just remove any duplicate factors and obtain a squarefree polynomial with the same zero set. $\square$

Lemma B.2 (Line interpolation). *Let $\mathcal{L}$ be a finite set of lines in $\mathbb{F}^D$. Then there is a squarefree $F \in \mathbb{F}[x_1, \ldots, x_D] \backslash \mathbb{F}$ such that $\cup\mathcal{L} \subset Z_{\mathbb{F}^D}(F)$ and $\deg(F) \leq D^2|\mathcal{L}|^{1/(D-1)}$.*

Proof. Let $d = D^2|\mathcal{L}|^{1/(D-1)}$. Arbitrarily choose d points on each line in $\mathcal{L}$, and call the resulting set of points S. Interpolate S using Lemma B.1, so that $S \subset Z(F)$ and

$$\deg(F) \leq D(d|\mathcal{L}|)^{1/D} < d.$$

Since the number of points on each line is larger than the degree of F, we get from Lemma A.7 that $\cup\mathcal{L} \subset Z(F)$. □

Lemma B.3 (Degree reduction). *Let $\mathcal{L}$ be a set of lines in $\mathbb{F}^3$ such that each line has at least A distinct intersection points with other lines of $\mathcal{L}$. Then there exists a squarefree $F \in \mathbb{F}[x, y, z]$ with $\cup\mathcal{L} \subset Z_{\mathbb{F}^3}(F)$ and $\deg(F) \lesssim A^{-1}|\mathcal{L}|$.*

Proof. Choose a random subset $\mathcal{L}' \subset \mathcal{L}$ by picking each line with probability p. Then the expected value of $|\mathcal{L}'|$ is $p|\mathcal{L}|$. Consider a line $\ell \in \mathcal{L}\backslash\mathcal{L}'$, which by assumption contains at least A intersection points. At each intersection point, an intersecting line is in $\mathcal{L}'$ with probability p. Thus the expected number of intersection points with $\mathcal{L}'$ on ℓ is at least pA.

For a large constant B and a small constant C, we have with positive probability[1] that $|\mathcal{L}'| < Bp|\mathcal{L}|$, as well as that on each $\ell \in \mathcal{L}\backslash\mathcal{L}'$ there are at least CpA intersection points with $\mathcal{L}'$. Now apply Lemma B.2 to get a polynomial F with $\cup\mathcal{L}' \subset Z(F)$ and $\deg(F) \leq 9|\mathcal{L}'|^{1/2} \leq 9B^{1/2}p^{1/2}|\mathcal{L}|^{1/2}$.

We want $CpA > 9B^{1/2}p^{1/2}|\mathcal{L}|^{1/2}$, so we choose $p = (9B^{-1}C^{-2})A^{-2}|\mathcal{L}|$. Then each line $\ell \in \mathcal{L}\backslash\mathcal{L}'$ contains more than $\deg(F)$ points on $Z(F)$, which implies by Lemma A.7 that $\ell \subset Z(F)$. So we get $\cup\mathcal{L} \subset Z(F)$ and also $\deg(F) \lesssim A^{-1}|\mathcal{L}|$.

The above only works when $p \leq 1$. Fortunately, when $A \leq |\mathcal{L}|^{1/2}$, we are done immediately by Lemma B.2. When $A > |\mathcal{L}|^{1/2}$, we can choose B large enough so that $9B^{-1}C^{-2} \leq 1$, which gives $p < 1$. □

B.2 Polynomial partitioning

Interpolation is based on the linear algebra fact that for any D points in $\mathbb{R}^D$, there is a hyperplane passing through all of them. Polynomial partitioning is based on a classic theorem from discrete geometry, which can be viewed as a generalization of this fact from linear algebra. It says that given any D sets in $\mathbb{R}^D$, there is a hyperplane that *bisects* each of these sets, in the sense that each halfspace of the hyperplane contains at most half the points of each set. If we take each set to be a single point, a bisecting hyperplane would have to contain each point, so this statement is a generalization of the linear algebra fact above. The proof, however, is less general, and relies on the Euclidean topology of real space.

Theorem B.4 (Ham-Sandwich). *Let $S_1, \ldots, S_m$ be finite subsets of $\mathbb{R}^E$, with $m \leq E$. There exists a linear polynomial $f \in \mathbb{R}[x_1, \ldots, \mathbb{R}_E]$ such that for each i*

$$|\{p \in S_i : f(p) > 0\}| \leq |S_i|/2 \quad \textit{and} \quad |\{p \in S_i : f(p) < 0\}| \leq |S_i|/2.$$

[1] One needs something like Chernoff's inequality to make this rigorous; we won't do that here, but the details can be found in [15, Lecture 12].

Interpolation worked by finding a hyperplane containing $V_{D,d}(S) \subset \mathbb{R}^{\binom{D+d}{D}}$, which gives a hypersurface containing $S \subset \mathbb{R}^D$. But we can also see that $\sum a_I x^I > 0$ in $\mathbb{R}^{\binom{D+d}{D}}$ if and only if $\sum a_I x^I > 0$ in $\mathbb{R}^D$, and similarly for $<$. Thus a hyperplane that bisects $V_{D,d}(S) \subset \mathbb{R}^{\binom{D+d}{D}}$ corresponds to a hypersurface that bisects $S \subset \mathbb{R}^D$, in the sense that each of the connected components of the complement of the hypersurface contains at most half the points of S. We get the following statement.

Corollary B.5 (Polynomial Ham-Sandwich)**.** *Let $T_1, \dots, T_m$ be finite subsets of $\mathbb{R}^D$. Then there exists a polynomial $F \in \mathbb{R}[x_1, \dots, x_D]$ of degree at most $Dm^{1/D}$ such that each connected component of $\mathbb{R}^D \backslash Z(F)$ contains at most $|T_i|/2$ points from each T_i.*

Proof. Choose d so that $\binom{D+d}{D} > m$; this is possible with $d = Dm^{1/D}$ (see the proof of Lemma B.1). Apply Theorem B.4 to the sets $V_{D,d}(T_i)$, which gives a bisecting linear polynomial $\sum a_I z_I$. Set $F := \sum a_I x^I$. On a connected component C of $\mathbb{R}^D \backslash Z(F)$, we have either $F > 0$ for all points in C or $F < 0$ for all points in C. Thus the image $V_{D,d}(C)$ lies entirely on one side of the hyperplane $\sum a_I x_I = 0$, and therefore contains at most half the points in each $V(T_i)$, which implies that C contains at most half the points in each T_i. $\square$

Using this corollary we can prove the polynomial partitioning theorem. The idea is simple: We repeatedly apply Corollary B.5 to bisect the points in each cell of the complement of the hypersurface that we have so far.

Theorem B.6 (Polynomial partitioning)**.** *Let $\mathcal{P} \subset \mathbb{R}^D$. For any integer r with $r \geq 1$ there exists a polynomial $F \in \mathbb{R}[x_1, \dots, x_D]$ of degree at most r such that $\mathbb{R}^D \backslash Z(F)$ has $O_D(r^D)$ connected components, each containing $O_D(|\mathcal{P}|/r^D)$ points of $\mathcal{P}$.*

Proof. Set $\mathcal{P}^0 := \mathcal{P}$. By Theorem B.4, there is a linear polynomial F_0 that bisects $\mathcal{P}^0$. Thus each of the two connected components of $\mathbb{R}^D \backslash Z(F_0)$ contains at most $|\mathcal{P}^0|/2$ points of $\mathcal{P}^0$; label these subsets of $\mathcal{P}^0$ as $\mathcal{P}^1_1$ and $\mathcal{P}^1_2$.

Set $\ell := \lfloor \log(r^D) \rfloor$ and repeat the following ℓ times. Given sets $\mathcal{P}^i_1, \dots, \mathcal{P}^i_{k_i}$, with $k_i = O(2^i)$, apply Corollary B.5 to get a polynomial F_i of degree $D2^{i/D}$, so that each connected component of $\mathbb{R}^D/Z(F_i)$ contains at most half the points of each $\mathcal{P}^i_j$. By Theorem A.10, the number of connected components of $\mathbb{R}^D/Z(F_i)$ is $O_D(\deg(F_i)^D) = O_D(2^i)$; let k_{i+1} be this number. Denote the subsets of $\mathcal{P}$ contained in each connected component of $\mathbb{R}^D/Z(F_i)$ by $\mathcal{P}^{i+1}_1, \dots, \mathcal{P}^{i+1}_{k_{i+1}}$.

After ℓ repetitions, set $F := F_0 \cdots F_\ell$. The degree of F is

$$\deg(F) = \sum_{i=0}^{\ell} D2^{i/D} = O_D(2^{\ell/D}) = O_D((2^{\log(r^D)})^{1/D}) = O_D(r).$$

By Theorem A.10, $\mathbb{R}^D \backslash Z(F)$ has $O_D(r^D)$ connected components. Since in each step the number of points of $\mathcal{P}$ in a connected component was divided by at least two, each connected component of $\mathbb{R}^D \backslash Z(F)$ contains at most $|\mathcal{P}|/2^\ell = O(|\mathcal{P}|/r^D)$ points of $|\mathcal{P}|$.

We are almost done, but the theorem claims that F has degree at most r rather than $O_D(r)$. If C_D is the constant in this $O_D(r)$, then we can run the above with $s = \lfloor r/C_D \rfloor$. This gives a polynomial of degree at most $C_D s = r$, such that $\mathbb{R}^D \backslash Z(F)$ has $O_D(s^D) = O_D(r^D)$ connected components, each containing $O(|\mathcal{P}|/s^D) = O_D(|\mathcal{P}|/r^D)$ points of $\mathcal{P}$. $\square$

B.3 Second partitioning polynomial

Lemma B.7 (Second partitioning polynomial). *Let $Z(F)$ be an irreducible surface in $\mathbb{R}^3$ with $\deg(F) = d$, and let $\mathcal{P}$ be a point set contained in $Z(F)$. Then for every $r \geq d$ there is a polynomial $G \in \mathbb{R}[x, y, z]$, coprime with F, such that $Z(F) \backslash Z(G)$ has $O(dr^2)$ connected components, each containing $O(|\mathcal{P}|/(dr^2))$ points of $\mathcal{P}$.*

(NOTE: Proof will be added later.)

Appendix C

Plücker coordinates and ruled surfaces

C.1 Plücker coordinates

(NOTE: Still under construction.)

We start with a very quick summary of projective space. We can define *projective space* $\mathbb{FP}^D$ as the set of all lines through the origin in $\mathbb{F}^{D+1}$. A point in $\mathbb{FP}^D$ can then be represented by coordinates $[x_0 : x_1 : \cdots : x_D]$, where $(x_0, x_1, \ldots, x_D)$ is any nonzero point on the corresponding line in $\mathbb{F}^{D+1}$. Two representations $[x_0 : x_1 : \cdots : x_D], [x_0' : x_1' : \cdots : x_D']$ define the same point if and only if the points $(x_0, x_1, \cdots, x_D), (x_0', x_1', \cdots, x_D')$ lie on the same line through the origin in $\mathbb{F}^{D+1}$, or equivalently if there is a $\lambda \in \mathbb{F}\backslash\{0\}$ such that $x_i = \lambda x_i'$ for all i.

The *affine space* $\mathbb{F}^D$ can be embedded in $\mathbb{FP}^D$ in many ways, but here we will take $(x_1, \ldots, x_D) \mapsto [1 : x_1 : \cdots : x_D]$ as the standard embedding. Then $\mathbb{FP}^D\backslash\mathbb{F}^D$ consists of the points of the form $[0 : y_1 : \cdots : y_D]$, which can be viewed as a copy of $\mathbb{FP}^{D-1}$. For instance, $\mathbb{FP}^3$ consists of $\mathbb{F}^3$ together with a copy of $\mathbb{FP}^2$, which is referred to as the plane at infinity.

We now introduce Plücker coordinates, which represent lines in $\mathbb{FP}^3$. Given two points $x = [x_0 : x_1 : x_2 : x_3], y = [y_0 : y_1 : y_2 : y_3]$, write ℓ_{xy} for the line through them (obviously a line has many such representations). The Plücker coordinates of ℓ_{xy} are given by

$$[p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}] \in \mathbb{FP}^5,$$

where

$$p_{ij} = x_i y_j - x_j y_i.$$

To make the formulas easier to remember, one can think of the p_{ij} as the determinants of the 2×2 submatrices of

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}.$$

The ordering of the p_{ij} is motivated by the following fact. Under the standard embedding $(x_1, x_2, x_3) \mapsto [1 : x_1 : x_2 : x_3]$, the line in $\mathbb{F}^3$ through two points $x, y \in \mathbb{F}^3$ is represented by

$$(p_{01}, p_{02}, p_{03}) = y - x, \qquad (p_{23}, p_{31}, p_{12}) = x \times y,$$

where $\times$ represents the cross product. In other words, the first three coordinates form the direction vector of the lines, and the last three coordinates are given by the cross product of the points x and y, which is the *moment vector* of the line. Conversely, these two vectors determine the line.

There are of course several things to check, which we won't do here. Mainly, one should check that if x, y and x', y' span the same line, then they lead to the same representations in Plücker coordinates.

Not every point in $\mathbb{FP}^5$ represents a line. One can show that the image of P consists of all points in $\mathbb{FP}^5$ satisfying

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0.$$

We won't prove this here, but one can justify in the direction-moment view that the Plücker coordinates of a line should satisfy this equation, because the direction of the line must be orthogonal to the moment, i.e. $(y - x) \cdot (x \times y) = 0$, which is exactly this equation. We call the zero set of this equation in $\mathbb{FP}^5$ the *Klein quadric* and denote it by $\mathcal{K}$. It is a four-dimensional irreducible projective variety with degree two.

Given two points

$$[p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}], \quad [p'_{01} : p'_{02} : p'_{03} : p'_{23} : p'_{31} : p'_{12}] \in \mathcal{K},$$

the corresponding lines in $\mathbb{FP}^3$ intersect if and only if

$$p_{01}p'_{23} + p_{02}p'_{31} + p_{03}p'_{12} + p'_{01}p_{23} + p'_{02}p_{31} + p'_{03}p_{12} = 0.$$

If we write the points as $[y - x : x \times y], [y' - x' : x' \times y']$, then this equation becomes

$$(y - x) \cdot (x' \times y') + (y' - x') \cdot (x \times y) = 0.$$

C.2 Ruled surfaces

Definition C.1 (Ruled surfaces)**.** Note that in these definitions we only consider irreducible surfaces.

- **Ruled surface**: An irreducible surface S in $\mathbb{CP}^3$ or $\mathbb{C}^3$ is a *ruled surface* if every point of S lies on a line that is contained in S;
- **Singly-ruled surface**: A ruled surface S in $\mathbb{CP}^3$ or $\mathbb{C}^3$ is *singly-ruled* if there is a curve $C \subset S$ such that every point in $S \backslash C$ lies on exactly one line contained in S;
- **Doubly-ruled surface**: A ruled surface S in $\mathbb{CP}^3$ or $\mathbb{C}^3$ is *doubly-ruled* if there is a curve $C \subset S$ such that every point in $S \backslash C$ lies on exactly two lines contained in S;
- **Cone**: A ruled surface S in $\mathbb{CP}^3$ or $\mathbb{C}^3$ is a *cone* if it is not a plane, and there exist a point p and a plane π such that for every point $q \in S \cap \pi$, the line through p and q is contained in S (p is called the *apex* of S).

Below we will show that every ruled surface is either singly-ruled, doubly-ruled, or a plane (which one could call "infinitely-ruled"). Note that a cone is a singly-ruled surface.

First we show that planes and cones are easy to distinguish because they have an *exceptional point*, which is a point contained in infinitely many lines contained in the surface. On a plane every point is exceptional, while on a cone there is a unique exceptional point (the apex), and there are no other possibilities.

Lemma C.2. *Let S in $\mathbb{CP}^3$ or $\mathbb{C}^3$ be a ruled surface. If some point $p \in S$ is contained in infinitely many lines that are contained in S, then S is a plane or a cone.*

Proof. By applying a projective transformation we can assume that p is the point $[0 : 0 : 0 : 1]$ at infinity (in the affine plane, this means that the surface contains infinitely many vertical lines). Let $\mathcal{F}$ be a homogeneous polynomial such that $S = Z_{\mathbb{CP}^3}(\mathcal{F})$. Expand $\mathcal{F}$ as

$$\mathcal{F}(w, x, y, z) = \sum \mathcal{F}_i(w, x, y)z^i.$$

We have infinitely many points $[w : x : y]$ such that $\mathcal{F}(w, x, y, z) = 0$ for all z, which implies that for these points $[w : x : y]$ we have $\mathcal{F}_i(w, x, y) = 0$ for all i. If at least two of the polynomials $\mathcal{F}_i$ are not identically zero, then by Bézout's inequality, they must have a common factor. However, this would contradict $\mathcal{F}$ being irreducible. Thus only one single $\mathcal{F}_i$ is not identically zero, and it must be $\mathcal{F}_0$, because if it was one of the others, it would again contradict the irreducibility of $\mathcal{F}$. Therefore, we have $\mathcal{F}(w, x, y, z) = \mathcal{F}_0(w, x, y)$, and $Z_{\mathbb{CP}^3}(\mathcal{F})$ consists exactly of all vertical lines through the curve $Z_{\mathbb{CP}^2}(\mathcal{F}_0)$ in the plane $z = 0$. Reversing the projective transformation, we get that S consists of all lines through p and the image of the curve $Z_{\mathbb{CP}^2}(\mathcal{F}_0)$ (which lies inside some plane). If $\mathcal{F}_0$ is linear, then S is a plane, and otherwise it is a cone.

If a ruled surface S in $\mathbb{C}^3$ has a point contained in infinitely many lines in S, then the same is true for the projective closure $\overline{S}$ of S. Thus $\overline{S}$ is a plane or a cone, which implies the same for S. $\square$

Next we show that a ruled surface cannot be "triply-ruled" (or more), unless it is a plane.

Lemma C.3. *Let $S \subset \mathbb{CP}^3$ be a ruled surface that is not a plane. Then there exists a one-dimensional subvariety $C \subset S$ such that any point $p \in S \backslash C$ lies on at most two lines that are contained in S.*

Proof. Let $Z(F)$ be the affine part of S. Let U be the subset of points in $Z(F)$ that lie on at least three lines in $Z(F)$. Let T_F be the polynomial vector from Theorem D.1. Then by part (b) of that theorem we have $\mathrm{T}_F(p) = 0$ for all points that lie on three or more lines contained in $Z(F)$, so we have $U \subset Z(F) \cap Z(T_F)$. By part ($c$), we have $\dim(Z(F) \cap Z(T_F)) \leq 1$, since otherwise $Z(F)$ would be a plane. Thus with the excluded curve $Z(F) \cap Z(T_F)$ the conclusion of the lemma holds for the affine part $Z(F)$. Let C be the union of the projective closure of $Z(F) \cap Z(T_F)$ with the curve at infinity of S. Then the conclusion of the lemma holds for the projective surface S. $\square$

This lemma does not yet imply that any non-plane ruled surface is singly-ruled or doubly-ruled, because a priori a surface could have many points on one line and many points on two lines. We now show that this cannot happen. The proof requires somewhat more advanced algebraic geometry, and we omit some details; a more rigorous version can be found in [35, Theorem 10].

Lemma C.4. *Every ruled surface in $\mathbb{CP}^3$ or $\mathbb{C}^3$ is either singly-ruled, doubly-ruled, or a plane.*

Proof. Let $S \subset \mathbb{CP}^3$ be a ruled surface that is not a plane, a cone, or a doubly-ruled surface. Let $F(S)$ be its Fano variety, which is the set of all points in the Klein quadric $\mathcal{K}$ corresponding to lines that are contained in S (it is a variety by [19, Example 6.19]). Then $F(S)$ is a subvariety of $\mathcal{K}$, and it is one-dimensional because S is not a plane. Define

$$X = \{(p, \ell) \in S \times F(S) : p \in \ell\}.$$

Then X is a two-dimensional irreducible projective variety. Let φ be the projection map $X \to S$, and ψ the projection map $X \to F(S)$; both are surjective.

If $F(S)$ had more than one one-dimensional component, then S would be doubly-ruled. Indeed, let V_1, V_2 be two one-dimensional irreducible components of $F(S)$. Then $C = \varphi(\psi^{-1}(V_1 \cap V_2))$ is a union of a finite set of lines, and any $p \in S \backslash C$ is contained in two lines contained in S. This implies that S is doubly-ruled. (The zero-dimensional components of $F(S)$ correspond to exceptional lines on S, though we will not prove that here.)

For a point $p_0 \in S$, $\varphi^{-1}(p_0)$ is the set of all (p_0, ℓ) with ℓ a line through p_0, so $|\varphi^{-1}(p_0)|$ is the number of lines through p_0. Then $C = \{p \in S : |\varphi^{-1}(p)| \geq 2\}$ is a subvariety of S; this follows from a standard theorem from algebraic geometry (see [19, Corollary 11.13]), which says that, given a regular map $\varphi : X \to Y$ between projective varieties, with X irreducible, then any set of the form $\{p \in Y : |\varphi^{-1}(p)| \geq k\}$ is a subvariety of Y. If C is two-dimensional, then S would be a doubly-ruled surface. Thus C is a subvariety of dimension at most one, and every point not in C is contained in exactly one line contained in S. This means that S is singly-ruled. The corresponding statement for affine varieties then follows. □

Next we work towards understanding doubly-ruled surfaces better. For this we need the following technical lemma.

Lemma C.5. *Let $S \subset \mathbb{CP}^3$ be an irreducible surface, and let ℓ be a line contained in S. Let V_ℓ be the union of all lines that are contained in S and that intersect ℓ. Then V_ℓ is a subvariety of S.*

Proof. We have $S = Z_{\mathbb{CP}^3}(F)$ for some homogeneous polynomial $F \in \mathbb{C}[x_0, x_1, x_2, x_3]$. A line through a point $q = [q_0 : q_1 : q_2 : q_3] \in \ell$ and a point $p = [p_0 : p_1 : p_2 : p_3] \notin \ell$ can be written as

$$\{[sq_0 + tp_0 : sq_1 + tp_1 : sq_2 + tp_2 : sq_3 + tp_3] : [s : t] \in \mathbb{CP}^1\}.$$

Then we can write V_ℓ more formally as

$$V_\ell = \{p \in \mathbb{CP}^3 : \exists q \in \ell \text{ such that } F(sq_0 + tp_0, sq_1 + tp_1, sq_2 + tp_2, sq_3 + tp_3) \equiv 0\}.$$

If we write $S = Z(F)$ and expand

$$F(sq_0 + tp_0, sq_1 + tp_1, sq_2 + tp_2, sq_3 + tp_3) = \sum F_i(p, q)s^i t^{d-i},$$

then

$$\begin{aligned} V_\ell &= \{p \in \mathbb{CP}^3 : \exists q \in \ell \text{ such that } \sum F_i(p, q)s^i t^{d-i} \equiv 0\}. \\ &= \{p \in \mathbb{CP}^3 : \exists q \in \ell \text{ such that } (p, q) \in Z(F_0, \dots, F_d)\} \\ &= \pi(Z_{\mathbb{CP}^3\times\mathbb{CP}^3}(F_1, \dots, F_d)), \end{aligned}$$

where $\pi : \mathbb{CP}^3 \times \mathbb{CP}^3 \to \mathbb{CP}^3$ is the standard projection that drops the fourth coordinate. By [19, Theorem 3.5], the projection of a projective variety is also a projective variety, so V_ℓ is a projective variety. □

We define an *exceptional line* of a ruled surface S to be a line contained in S that has infinitely many intersection points with other lines contained in S. Unlike exceptional points, exceptional lines can show up in many different ways. Fortunately, the following lemma shows that there can be at most two, unless the surface is doubly-ruled, in which case the subsequent lemma shows that there are exactly three.

Lemma C.6. *Let S in $\mathbb{CP}^3$ or $\mathbb{C}^3$ be a ruled surface that is not a plane or a doubly-ruled surface. Then S has at most two exceptional lines.*

Proof. The statement for $\mathbb{CP}^3$ directly implies the statement for $\mathbb{C}^3$, so consider $S \subset \mathbb{CP}^3$. Let ℓ be an exceptional line of S, and let V_ℓ be the union of all lines contained in S that intersect ℓ. By Lemma C.5, V_ℓ is a subvariety of S. If V_ℓ were one-dimensional, then it would consist of only finitely many lines, contradicting the assumption. Thus V_ℓ is two-dimensional, and since S is an irreducible two-dimensional variety, it follows that $V_\ell = S$. This implies that S is entirely covered by lines passing through ℓ, and that every line contained in S passes through ℓ.

Suppose that S has three exceptional lines ℓ_1, ℓ_2, ℓ_3. By the argument above, a point $p \in S$ must lie on a line intersecting ℓ_1, and this line must also intersect ℓ_2 and ℓ_3. In other words, every point of S lies on a line intersecting ℓ_1, ℓ_2, ℓ_3.

By line interpolation, there is a quadric projective surface T containing ℓ_1, ℓ_2, ℓ_3. By Bézout's inequality (specifically Lemma A.7), any line that intersects ℓ_1, ℓ_2, ℓ_3 must be contained in T. Since every point of S is on such a line, it follows that $S \subset T$. If T is irreducible, then $S = T$ is a doubly-ruled surface. If T is reducible, it is the union of two planes, and S must be one of these planes. □

As said, we now show that a doubly-ruled surface has three exceptional lines, and is in fact determined by these lines.

Lemma C.7. *Let $S \subset \mathbb{CP}^3$ be an irreducible surface. Then the following are equivalent.*
(a) *S is a doubly-ruled surface;*
(b) *S is a smooth quadric;*
(c) *there are three non-coplanar lines in $\mathbb{C}^3$ such that S is the union of all lines intersecting these three lines.*

Proof. Suppose that S is doubly-ruled. By Lemmas C.2 and C.6, S has no exceptional point and at most two exceptional lines, so we can ignore these in the proof. By definition, there is a curve $C \subset S$ such that any point outside C lies in exactly two lines on S; we will ignore lines that are contained in C.

Let $\ell \subset S$ be any line. Any point in $\ell \backslash C$ lies on another line contained in S. Let ℓ_1, ℓ_2, ℓ_3 be three lines that intersect $\ell \backslash C$ in distinct points; on these lines too all but finitely many points are contained in another line on S. Moreover, by Lemma C.5, each point of S lies on a line that intersects ℓ_i, for each i. Therefore, any point p in $\ell_1 \backslash C$ must lie on a line that intersects ℓ_2 and on a line that intersects ℓ_3. Since $p \notin C$, it is not a triple point, so there must be a single line through p that intersects both ℓ_2 and ℓ_3.

Apply line interpolation to ℓ_1, ℓ_2, ℓ_3 to get a quadric surface Q that contains all three lines. By the above, there are infinitely many lines in S that intersect Q in at least three points, so these lines must also be contained in Q. It follows that $S = Q$ and that S is the union of all lines intersecting ℓ_1, ℓ_2, ℓ_3. This proves that (c) follow from (a). The same interpolation argument also shows that (c) implies (b), except for the quadric being smooth, but that follows from the classification of quadrics below.

Finally, we prove that (b) implies (a), by giving a classification of quadric that makes this obvious. A quadric is defined by a quadratic homogeneous form, which can be diagonalized and normalized. This means that there is a projective transformation of $\mathbb{CP}^3$ that turns the quadric into the zero set of one of the equations (corresponding to the matrix of the quadratic form having rank 1, 2, 3, or 4)

$$x_0^2 = 0, \quad x_0^2 + x_1^2 = 0, \quad x_0^2 + x_1^2 + x_2^2 = 0, \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0.$$

The zero set of the first equation is a plane, which is not really a quadric. The zero set of the second equation is the union of two planes, and thus not smooth (the intersection of the two planes is singular). The zero set of the third equation is a cone, which is singular (the singular point is $[0 : 0 : 0 : 1]$). Finally, the last equation represents all smooth quadrics. In particular, all smooth quadrics must be projectively equivalent to the smooth quadric $x_0x_1 = x_2x_3$. The affine equation of this quadric is $xy = z$, which is doubly-ruled: Take for instance the family of lines $\ell_\alpha = \{(t, \alpha, \alpha t) : t \in \mathbb{C}\}$ and the family $\ell_\beta = \{(\beta, t, \beta t) : t \in \mathbb{C}\}$. □

Finally, in the proof of the Guth-Katz Theorem A we need the following observation, which is fairly intuitive given the lemmas above, but requires a bit of care to make rigorous.

Lemma C.8. *Let S be a singly-ruled surface in $\mathbb{CP}^3$ or $\mathbb{C}^3$ that is not a cone, let $\ell \subset S$ be a non-exceptional line, and let π be a plane that does not contain ℓ. Then, for every open neighborhood B of $\ell \cap \pi$, there is a line contained in S that intersects $B \cap \pi$ in a point other than $\ell \cap \pi$.*

Proof. Let p be the point on the Klein quadric corresponding to ℓ. From the proof of Lemma C.4, we know that p lies in a one-dimensional irreducible component of the Fano variety $F(S)$. This component is a continuous curve, which implies that for every open neighborhood of p in $F(S)$, there is another point of $F(S)$, corresponding to a line ℓ'. We can assume that ℓ' is not contained in π, since S can only have finitely many lines in common with π.

The points on ℓ' can be found by solving a linear system of equations, consisting of the equations that define the Plücker coordinates; the coefficients of this system are linear in terms of the Plücker coordinates. This system obviously has a one-dimensional set of solutions (namely ℓ'), and combining it with the defining equation of the plane π gives a system with a unique solution (otherwise ℓ' would be contained in π), which is the point $\ell' \cap \pi$. By inverting the matrix of this system, we see that the coordinates of $\ell' \cap \pi$ are given by a continuous function φ of the Plücker coordinates of ℓ'.

Given an open neighborhood B of $\ell \cap \pi$, $\varphi^{-1}(B)$ is an open neighborhood of p because φ is continuous. Then there is a point $p' \in \varphi^{-1}(B) \cap F(S)$. Since $\varphi(p') \in B$, the line ℓ' corresponding to p' intersects π in a point that lies in B; since $p' \in F(S)$, ℓ' is contained in S. This completes the proof. $\square$

Appendix D

Detection polynomials

In this section we introduce the "detection polynomials" that play an important role in Chapters 3 through 6. Roughly speaking, these polynomials "detect" special points on a surface $Z(F)$ in $\mathbb{C}^3$ or $\mathbb{R}^3$. The most familiar is the *gradient* ∇F, which detects singular points of the surface; for this polynomial (actually, a vector of polynomials) we have little to prove. The second polynomial (also a vector) is what we refer to here as the *triple point polynomial* T_F, which detects intersection points of at least three lines contained in the surface. Such points are either singular, or they are *flat* points of the surface. Finally, we will meet the *flecnode polynomial* Fl_F (actually a polynomial), which detects points that lie on lines contained in the surface.

The ideas behind these polynomials come from differential geometry, though here we will introduce everything in an algebraic way, relying on some basic linear algebra. For a function $F(x_1, x_2, x_3)$, we write F_{x_i} for the partial derivative of F with respect to x_i. We will use vector notation somewhat loosely, for instance leaving it to context whether a letter represents a vector or a scalar, and whether $\cdot$ denotes scalar multiplication or dot product. Note that we are only using the standard inner product $u \cdot v = \sum u_i \cdot v_i$, not the Hermitian inner product that makes $\mathbb{C}^D$ into an inner product space (it involves a conjugate, so is not even a polynomial function). Notions like orthogonality are still well-defined.
(NOTE: I plan to still add some more explanation to this chapter. A few things are not completely proved.)

D.1 Detection polynomials

We fix a polynomial $F \in \mathbb{C}[x_1, x_2, x_3]$. Consider the following polynomials in six variables, of which the first three form a point $p \in \mathbb{C}^3$, and the last three form a direction $v \in \mathbb{C}^3$, considered as a vector based at p.

$$F_1(p, v) = \sum_i F_{x_i} v_i = F_{x_1}(p) \cdot v_1 + F_{x_2}(p) \cdot v_2 + F_{x_3}(p) \cdot v_3;$$

$$F_2(p, v) = \sum_{i,j} F_{x_i x_j} v_i v_j = F_{x_1 x_1}(p) \cdot v_1^2 + F_{x_1 x_2}(p) \cdot v_1 v_2 + \cdots + F_{x_3 x_3}(p) \cdot v_3^2;$$

$$F_3(p, v) = \sum_{i,j,k} F_{x_i x_j x_k} v_i v_j v_k = F_{x_1 x_1 x_1}(p) \cdot v_1^3 + F_{x_1 x_1 x_2}(p) \cdot v_1^2 v_2 + \cdots + F_{x_3 x_3 x_3}(p) \cdot v_3^3.$$

Around a point p, the Taylor expansion of F at a $p+v$ is

$$F(p+v) = F(p) + F_1(p,v) + F_2(p,v) + F_3(p,v) + \cdots .$$

Also, at a fixed point p, the set of all points $p+v$ such that $F_1(p,v) = 0$ is the *tangent plane* to $Z(F)$ at p. Indeed, we can write $F_1(p,v) = (\nabla F)(p) \cdot v$, and a direction v is a tangent direction (i.e., $p+v$ lies in the tangent plane) if and only if v is orthogonal to the gradient vector.

We can summarize the three "detection polynomials" as follows. Given $F \in \mathbb{C}[x,y,z]$, there exist polynomial vectors $\nabla F, \mathsf{T}_F \in (\mathbb{C}[x,y,z])^3$ and a polynomial $\mathsf{Fl}_F \in \mathbb{C}[x,y,z]$ such that the following hold for any $p \in Z(F)$:

- $(\nabla F)(p) = 0$ if and only if $F_1(p,v)$ vanishes for all directions v; in other words, there is no well-defined tangent plane to $Z(F)$ at p;
- $\mathsf{T}_F(p) = 0$ if and only if $F_2(p,v) = 0$ for all v with $F_1(p,v) = 0$; i.e., F vanishes to second order in each direction that lies on the tangent plane at p;
- $\mathsf{Fl}_F(p) = 0$ if and only if there is a v such that $F_i(p,v) = 0$ for $i = 1,2,3$; this means that for some direction in the tangent plane at p, F vanishes to third order.

The gradient should be familiar, and we don't need to prove its existence. For the second and third, aside from their existence and degree, we need to establish what happens when they vanish on $Z(F)$; for T_F this should mean that $Z(F)$ is a plane, while for Fl_F it should mean that $Z(F)$ is a ruled surface.

D.2 The triple point polynomial

The role of T_F is similar to that of the *second fundamental form*, but that is a differential form that gives a quadratic form on each tangent plane, and which vanishes whenever a point is singular or *flat*. We need a polynomial variant (Guth and Katz [17] called it an "algebraic second fundamental form").

Theorem D.1. *Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. For $F \in \mathbb{F}[x,y,z]$ there exists a polynomial vector $\mathsf{T}_F \in (\mathbb{F}[x,y,z])^3$ with the following properties.*

a) $\deg(\mathsf{T}_F) \le 3\deg(F)$*;*

b) If $p \in Z(F)$ is an intersection point of three lines contained in $Z(F)$, then $\mathsf{T}_F(p) = 0$;

c) If $Z(G) \subset Z(F) \cap Z(\mathsf{T}_F)$, then $Z(G)$ contains a plane.

Proof. Consider a fixed p for which $(\nabla F)(p) \neq 0$. If $F_1(p,v) = 0$ for some v, then we can think of v as a vector in the tangent plane at p. We could choose any three distinct vectors $\{u,v,w\}$ in this tangent plane, and set $T_F(p) = (F_2(p,u), F_2(p,v), F_2(p,w))$. Below we'll give an argument that shows that this implies that F_2 vanishes on all of the tangent plane, so this definition would do the job at p. However, we need this choice to be systematic, and we need it to depend polynomially on p. One way to do this is to choose the three vectors $(\nabla F)(p) \times e_i$, where e_i is the ith standard basis vector. Each vector $\nabla F \times e_i$ is orthogonal to ∇F, so lies in the tangent plane. Thus we let

$$\mathrm{T}_F(p) = (F_2(p, \nabla F \times e_1), F_2(p, \nabla F \times e_2), F_2(p, \nabla F \times e_3)).$$

It is easy to see that $\deg(\mathrm{T}_F) \leq 3\deg(F)$.

We now show property b). Suppose three lines are contained in $Z(F)$ and intersect at p. If $(\nabla F)(p) = 0$, then $\mathrm{T}_F(p) = 0$ is clear from the definition. Otherwise, $Z(F)$ has a well-defined tangent plane at p. The three lines must lie in the tangent plane, and $F_2(p, v)$ must vanish on them. For any point w in the tangent plane that is not on one of these three lines, choose a line through w and not through p. Then F_2 vanishes on three points on this line, so by Bézout's Inequality (Theorem A.4), F_2 vanishes on the entire line. Therefore, F_2 vanishes on w, and similarly on the entire tangent plane, so $\mathrm{T}_F(p) = 0$.

Next we show that $\mathrm{T}_F(p) = 0$ if and only if $F_2(p, v) = 0$ for all v with $F_1(p, v) = 0$ (assuming that $(\nabla F)(p) \neq 0$). It is clear that if $F_2(p, v) = 0$ for all v with $F_1(p, v) = 0$, then this also holds for the $\nabla F \times e_i$, so $\mathrm{T}_F(p) = 0$. If $\mathrm{T}_F(p) = 0$, then we have $F_2(p, t \cdot (\nabla F \times e_i)) = 0$ for every $t \in \mathbb{F}$, because F_2 is homogeneous. So $F_2(p, v)$ vanishes on three lines in the tangent plane, and by the same argument as above it vanishes on the entire tangent plane.[1]

Finally, we show that if $Z(G)$ is a two-dimensional irreducible component of $Z(F)$ on which T_F vanishes, then $Z(G)$ must be a plane. This requires a few more tools from differential geometry. Let $p \in Z(G) \subset Z(F) \cap Z(\mathrm{T}_F)$ be a nonsingular point of $Z(G)$. Using the implicit function theorem, we can parametrize the surface around p by a differentiable map $q(u, v) = (x(u, v), y(u, v), z(u, v))$. Then we have $F(x(u, v), y(u, v), z(u, v)) = 0$, and differentiating this with respect to u gives $F_x x_u + F_y y_u + F_z z_u = 0$. Differentiating around $q(u, v)$ with respect to u again gives

$$\begin{aligned} 0 &= F_x x_{uu} + F_y y_{uu} + F_z z_{uu} + F_{xx} x_u^2 + F_{xy} x_u y_u + \cdots \\ &= \nabla F \cdot q_{uu} + F_2(q, q_u) \\ &= \nabla F \cdot q_{uu}, \end{aligned}$$

using that $F_2(q, q_u) = 0$ because q_u lies in the tangent plane at q. We also have $\nabla F \cdot q_u = 0$, and differentiating this and using the previous equation gives

$$0 = \nabla F \cdot q_{uu} + (\nabla F)_u \cdot q_u = (\nabla F)_u \cdot q_u.$$

We can similarly get $(\nabla F)_u \cdot q_v = (\nabla F)_v \cdot q_u = (\nabla F)_v \cdot q_v = 0$. Since q_u and q_v span the tangent plane, this implies that $(\nabla F)_u$ and $(\nabla F)_v$ are orthogonal to the tangent plane, and thus parallel to ∇F. This implies that the direction of ∇F does not change, which means that $Z(F)$, and therefore $Z(G)$, coincides with a plane around p. Since $Z(G)$ is algebraic and irreducible, it must equal that plane. $\square$

[1] There is an unpleasant exception to this when one of the $\nabla F \times e_i$ is the zero vector, which happens if ∇F is parallel to e_i. But this only happens for p in a one-dimensional subset of $Z(F)$, so it follows by continuity that the property still holds at these p.

D.3 The flecnode polynomial

A *flecnode* on a surface $Z(F) \subset \mathbb{C}^3$ is a point that has a tangent line on which F vanishes to third degree.

Lemma D.2 (Flecnode polynomial). *For any $F \in \mathbb{C}[x, y, z]$, there exists a polynomial $\mathrm{Fl}_F \in \mathbb{C}[x, y, z]$ such that:*

a) $\deg(\mathrm{Fl}_F) \lesssim \deg(F)$*;*

b) If $p \in Z(F)$ is contained in a line that is contained in $Z(F)$, then $\mathrm{Fl}_F(p) = 0$;

c) If F is irreducible and $Z(F) \subset Z(\mathrm{Fl}_F)$, then $Z(F)$ is a ruled surface.

Proof. Proving the existence of Fl_F is fairly simple, although the calcution would be lengthy, so we merely argue that it is possible. We want to take the equations $F_i(p, v) = 0$ and eliminate the variables v_1, v_2, v_3. Assume p is nonsingular; then one of the derivatives $F_x(p), F_y(p), F_z(p)$ is nonzero, and we can assume it is $F_x(p)$. The first equation is linear, so we can use it to eliminate v_1 from the second and third equations. Then the second equation becomes a homogeneous quadratic equation in v_2, v_3, so we can factor it and get two homogeneous linear expressions for v_2 in terms of v_3. We plug each of these into the third equation, which will result in a homogeneous equation in v_3. This is v_3^3 multiplied by a polynomial in p, and that polynomial is $\mathrm{Fl}_F(p)$. Indeed, this polynomial is zero at p if and only if there are v_1, v_2, v_3 such that $F_i(p, v) = 0$ for $i = 1, 2, 3$. It should be clear that the degree of Fl_F is at most a small constant times the degree of F.

Property (b) is easy to prove. If $p \in \ell \subset Z(F)$ for a line ℓ, and v is a direction vector for ℓ, then $F_i(p, v) = 0$ for each i, which implies that $\mathrm{Fl}_F(p) = 0$.

Property (c) is harder. Let $p \in Z(F)$ be a nonsingular point, and let $\ell_v = \{p + tv : t \in \mathbb{C}\}$ be a line through p with direction vector v. The line ℓ_v is a tangent line at q if and only if $F_1(q, v) = 0$, and it is a double tangent line if and only if $F_1(q, v) = F_2(q, v) = 0$. The double tangent directions form a vector field on $Z(F)$, and this can be integrated to give *asymptotic curves* of the surface. These are curves $q(t) = (q_1(t), q_2(t), q_3(t))$, for t in some open set $B \subset \mathbb{C}$, such that for all $t \in B$

$$F(q) = 0, \quad F_1(q(t), q'(t)) = 0, \quad \text{and} \quad F_2(q(t), q'(t)) = 0.$$

Now suppose Fl_F vanishes on $Z(F)$. This means that we also have $F_3(q, q') = 0$. To summarize, we have (writing $x_1 = x, x_2 = y, x_3 = z$ for convenience)

$$0 = \sum_i F_{x_i} q_i' = \sum_{i,j} F_{x_i x_j} q_i' q_j' = \sum_{i,j,k} F_{x_i x_j x_k} q_i' q_j' q_k'.$$

Differentiating $0 = F_1(q, q')$ with respect to t gives

$$0 = \frac{\partial}{\partial t} \sum_i F_{x_i} q_i' = \sum_i F_{x_i} q_i'' + \sum_{i,j} F_{x_i x_j} q_i' q_j' = \sum_i F_{x_i} q_i'' = \nabla F \cdot q'',$$

where we used $F_2(q, q') = 0$. Then differentiating $0 = F_2(q, q')$ gives

$$0 = \frac{\partial}{\partial t} \sum_{i,j} F_{x_i x_j} q_i' q_j' = \sum_{i,j} F_{x_i x_j} (q_i'' q_j' + q_i' q_j'') + \sum_{ijk} F_{x_i x_j x_k} q_i' q_j' q_k' = 2 \sum_{i,j} F_{x_i x_j} q_i' q_j'', \quad \text{(D.1)}$$

using $F_3(q, q') = 0$. Finally, we differentiate $0 = \nabla F \cdot q''$ and use (D.1) to get

$$0 = \frac{\partial}{\partial t} \sum_i F_{x_i} q_i'' = \sum_i F_{x_i} q_i''' + 2 \sum_{i,j} F_{x_i x_j} q_i'' q_j' = \sum_i F_{x_i} q_i''' = \nabla F \cdot q'''.$$

Therefore, we have shown that ∇F is orthogonal to q', q'', and q'''. This implies that the torsion of the curve $q(t)$ is zero, so the curve lies in a plane. Then, inside the plane, we still have q', q'' orthogonal to the projection of ∇F onto the plane, so they are parallel vectors. This means that the curvature of the curve $q(t)$ is zero, which finally means that $q(t)$ is a line. $\square$

(NOTE: I'm not really sure that using torsion and curvature is justified over $\mathbb{C}$, I can't find a reference for it. Otherwise I need to find a direct proof.)

Appendix E

Various other tools

E.1 Combinatorial bounds

Lemma E.1. *Let G be a bipartite graph with vertex set $A \cup B$ and edge set E. Assume that G contains no copy of the complete bipartite graph $K_{s,t}$ (with the s vertices in A and the t vertices in B. Then*

$$|E| \lesssim_{s,t} |A|^s + |B|.$$

E.2 Pruning

Lemma E.2 (Pruning)**.** *Let G be a graph with vertex set V and edge set E. Then there is an induced subgraph G' such that G' has at least $|E|/2$ edges, and every vertex of G' is connected to at least $|E|/(2|V|)$ other vertices of G'.*

Proof. ... $\square$

Lemma E.3 (Pruning)**.** *Let G be a bipartite graph with vertex sets A and B, such that B has no isolated vertices. Then there are nonempty $A' \subset A, B' \subset B$ such that in the induced subgraph G' on $A' \cup B'$, every $a \in A'$ has $\deg_{G'}(a) \geq |B|/(2|A|)$, and every $b \in B'$ has $\deg_{G'}(b) = \deg_G(b)$.*

Proof. Set $A' := A, B' := B$ and repeat the following step until it is no longer possible. If a vertex in A' has degree less than $|B|/(2|A|)$, remove it from A' and also remove all its neighbors from B'. Note that this step preserves the property $\deg_{G'}(b) = \deg_G(b)$ for all $b \in B'$. In each step we remove exactly one vertex of A, so we are done in at most $|A|$ steps, and altogether we remove at most $|B|/2$ vertices from B. It follows that the final B' is nonempty, and because B has no isolated vertices, A' is also nonempty. Any vertex remaining in A' must have degree at least $|B|/(2|A|)$. $\square$

E.3 Inequalities

(Note: Cauchy-Schwarz, Hölder)

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