Advanced Discrete Mathematics 2013 – Problem Set 9

You can hand in one of the star problems before 3pm on Monday May 6th.

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1. Prove that a real symmetric matrix M is positive semidefinite ($x^T M x \geq 0$ for all x) if and only if all its eigenvalues are nonnegative, if and only if $M = U^T U$ for some matrix U with $\mathrm{rk}(U) = \mathrm{rk}(M)$.

2. Prove that the largest eigenvalue λ_1 of a graph G equals the maximum value of $x^T A_G x$ over all $x \in \mathbb{R}^{|V(G)|}$ with $x^T x = 1$.

3. Can you get a set of equiangular lines from the Petersen graph? Is it interesting?

4. Reprove Erdős-Ko-Rado with probability as follows. Choose a random set $A \in [n]^{(k)}$ by picking a random permutation of $[n]$, placing the numbers along a circle in this order, picking a random starting point s, and then taking A to be s and the next $k-1$ numbers along the circle. Prove that this is equivalent to picking A uniformly, and that the probability that A is from the intersecting set system is $\leq k/n$.

*5. Use eigenvalues to prove that if a graph contains no $K_{2,s}$, then $e \leq \frac{1}{2}\sqrt{s}n^{3/2}$.
Hint: Consider $v^t A^2 v$ with v an eigenvector for λ_1.

*6. Use eigenvalues to reprove the Graham-Pollak theorem, which says that the complete graph cannot be decomposed into $\leq n-2$ complete bipartite graphs.

*7. **Chromatic Number and Eigenvalues**
Recall that the *chromatic number* $\chi(G)$ of a graph G is the smallest number of colors that we can color the vertices of G with so that adjacent vertices have different colors.
   (a) Prove that if G is regular, then $\chi(G) \geq 1 + \frac{\lambda_1}{-\lambda_n}$.
   (b) Prove that if H is a subgraph of G, then the largest eigenvalue of H is no larger than the largest eigenvalue of G.
   (c) Use induction and (b) to prove that $\chi(G) \leq 1 + \lambda_1$.
