Advanced Discrete Mathematics 2013 – Problem Set 8

You can hand in one of the star problems before 3pm on Monday April 29th.

1. Show that if G has ≥ 4 vertices and every two of its vertices have exactly one common neighbor, but there is no vertex adjacent to all the others, then G must be regular.
   (This is part of the proof of the windmill theorem.)

2. Find two non-isomorphic graphs that have the same spectrum, with one connected and the other one not connected. Conclude that connectedness cannot always be determined from the spectrum.
   Hint: Take a C_4 and add an isolated point.

3. (a) Show that if t is the number of triangles K_3 in a graph, then
   $$6t = \sum_{i=1}^{n} \lambda_i^3.$$
   (b) Use this to (re)prove that if a graph has $> n^2/4$ edges, then it contains a K_3.
   Hint: Use $\lambda_1 \geq d_{avg}$ and the fact that if $\sum x_i^2 < y^2$, then $|\sum x_i^3| < |y^3|$.

4. (a) The *line graph* $L(G)$ of a graph G has as vertices the edges of G, with an edge between two vertices of $L(G)$ if the corresponding edges in G touch at a vertex.
   For a connected k-regular graph G with
   $$\text{Spec}(G) = (k)^1 (\lambda)^{m(\lambda)} \cdots (\omega)^{m(\omega)},$$
   show that
   $$\text{Spec}(L(G)) = (2k-2)^1 (\lambda + k - 2)^{m(\lambda)} \cdots (\omega + k - 2)^{m(\omega)} (-2)^{|E(G)|-|V(G)|}.$$
   Hint: Let B be the *incidence matrix* of A. Then BB^T and B^TB have mostly the same eigenvalues, and they have something to do with A_G and $A_{L(G)}$.
   (b) The *complement* $\overline{G}$ of G has the same vertices, but two vertices have an edge in $\overline{G}$ if and only if they do not have an edge in G.
   For a connected regular graph G with spectrum as in (a), show that
   $$\text{Spec}(\overline{G}) = (n-k-1)^1 (-\lambda - 1)^{m(\lambda)} \cdots (-\omega - 1)^{m(\omega)}.$$
   (c) Give the spectrum of $\overline{L(K_n)}$. Write it out for $\overline{L(K_5)}$ and $\overline{L(K_8)}$ (the first should be familiar; the second will be used in the next lecture).

*5. Prove that the Petersen graph does not have a Hamilton cycle.
   Hint: Think of it as a decomposition.

*6. Reprove the windmill theorem without using linear algebra, using the following outline. We've already proved without linear algebra that G is k-regular and $n = k^2 - k + 1$. Let $f(m)$ be the number of m-walks from a fixed vertex to itself. Then
   $$f(m) = (k-1)f(m-2) + k^{m-2}.$$
   Pick a prime p dividing $k - 1$. Then it follows from the formula for f that the total number of closed p-walks is $\equiv 1 \bmod p$, which gives a contradiction.