Advanced Discrete Mathematics 2013 – Problem Set 7

You can hand in one of the star problems before 3pm on Monday April 22nd.

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1. Prove the following facts about eigenvalues of graphs.
   (a) If G has at least one edge, then it has a negative eigenvalue.
   (b) Suppose G is connected. If an eigenvector of G is real and nonnegative (each entry is ≥ 0), then it is positive (each entry is > 0).
   (c) If an eigenvalue of a graph is in $\mathbb{Q}$, then it is in $\mathbb{Z}$.
   Hint: The characteristic polynomial $\det(A - xI)$ has integer coefficients.

2. Determine the spectrum of P_n, the path with n vertices and $n-1$ edges.
   Hint: If an eigenvector of a cycle has value 0 at a vertex, then you can remove that vertex. None of the eigenvectors of the cycle from class have a value 0, but if the cycle is even, then there are eigenvalues with multiplicity 2. So any linear combination of two eigenvectors for such an eigenvalue will be an eigenvector; some of these will have a value 0.

3. Prove that if G is d-regular, then the multiplicity of the largest eigenvalue λ_1 equals the number of connected components of G.
   Hint: Use the equality $\lambda x_u = \sum_{v \in N(u)} x_v$ for eigenvectors $(x_v)_{v \in V(G)}$.

4. Prove that if G is connected, then the diameter of G is strictly less than its number of distinct eigenvalues.

*5. (a) The *line graph* $L(G)$ of a graph G has as vertices the edges of G, with an edge between two vertices of $L(G)$ if the corresponding edges in G touch at a vertex.
   For a regular graph G, give the spectrum of $L(G)$ in terms of the spectrum of G.
   Hint: Let B be the *incidence matrix* of A. Then BB^T and B^TB have mostly the same eigenvalues, and they have something to do with A_G and $A_{L(G)}$.
   (b) The *complement* $\overline{G}$ of G has the same vertices, but two vertices have an edge in $\overline{G}$ if and only if they do not have an edge in G.
   For a regular graph G, give the spectrum of $\overline{G}$ in terms of the spectrum of G.
   (c) Use (a) and (b) (and nothing else!) to find the spectrum of the Petersen graph.

*6. Find the spectrum of the *cube graph* Q_n, which has vertices all subsets of $[n]$, with S and T connected by an edge if $|S \cap T| = 1$. (Equivalently, the vertices are n-dimensional 01-vectors, with two of them connected if they differ in one entry.)
