

Advanced Discrete Mathematics 2013 – Problem Set 7 – Solutions

1. Prove the following facts about eigenvalues of graphs.

(a) If G has at least one edge, then it has a negative eigenvalue.

The adjacency matrix A is not the zero matrix, so has some nonzero eigenvalues. On the other hand, we have

$$0 = \text{Tr}(A) = \sum_{i=1}^n \lambda_i,$$

so there must be a negative eigenvalue (as well as a positive one).

(b) Suppose G is connected. If an eigenvector of G is real and nonnegative (each entry is ≥ 0), then it is positive (each entry is > 0).

The corresponding eigenvalue λ cannot be 0, since then $Ax = 0$ and $x \geq 0$ would imply $x = 0$.

So for each entry x_u of an eigenvector $x = (x_v)_{v \in V(G)}$ we have

$$x_u = \frac{1}{\lambda} \sum_{v \in N(u)} x_v.$$

If $x_u = 0$, then we must have all $x_v = 0$ for $v \in N(u)$. Repeating this for the neighbors of the x_v , etc., we get that all entries of x are zero, because the graph is connected so we can reach every vertex this way. So $x = 0$, contradiction.

Note: If the graph is not connected, we can clearly create eigenvectors that are nonnegative on one component and 0 on some other component. If the eigenvector is not real, we could still make the same statement if we define $a + bi \geq 0$ to mean $a \geq 0$ and $b \geq 0$. The proof would be the same because the equation above would split into a real and an imaginary equation (using that λ is real).

(c) If an eigenvalue of a graph is in \mathbb{Q} , then it is in \mathbb{Z} .

The characteristic polynomial $f(x) = \det(A - xI)$ clearly has integer coefficients, and the coefficient of its leading term is 1. Then it is a well-known fact (sometimes called the Rational Root Theorem) that if a root is rational, then it is an integer. The proof went like this: Let p/q be a rational root, with p and q coprime. Let d be the degree of f . Then $0 = q^d f(p/q)$ is an integer equation, and q divides all terms except the first, which is p^d . That's not possible unless $q = 1$.

2. Determine the spectrum of P_n , the path with n vertices and $n - 1$ edges.

For an eigenvector x of the cycle $c_0c_1 \cdots c_{2n+1}$ with eigenvalue λ , we have $\lambda x_k = x_{k-1} + x_{k+1}$. So if $x_{k+1} = 0$, ignoring the vertex c_{k+1} would not break this equation (or the other one involving x_{k+1}), ie the subgraph obtained by removing this vertex has the same eigenvalue, and removing the 0 entry from x gives the eigenvector.

Unfortunately the usual eigenvectors of C_n do not have 0 entries. But we can get around that by taking an eigenvalue of multiplicity 2, and subtracting two eigenvectors from each other that are equal in some entry. An even cycle has such an eigenvalue (because of the symmetry of the cosine), but removing a vertex would only give an odd path. To get around this, we can take an even cycle that is twice as long as the path we want, and then remove two opposite vertices.

The eigenvector of C_{2n+2} corresponding to $\lambda_l = \omega^l + \omega^{-l}$ (with $\omega = e^{2\pi i/(2n+2)}$) is

$$u_l = (1, \omega^l, \omega^{2l}, \dots, \omega^{(n-1)l}).$$

We have $\lambda_l = \lambda_{-l}$, so this gives an eigenvalue with multiplicity 2, hence any linear combination of the eigenvectors u_l and u_{-l} gives another eigenvector for λ_l . Now u_l and u_{-l} both have first entry 1 and $(n+2)$ -th entry $\omega^{(n+1)l} = e^{\pi i} = \omega^{-(n+1)l}$. Therefore $v = u_l - u_{-l}$ has a 0 in entries 1 and $n+1$, and it is also an eigenvector for λ_l .

Now remove the two vertices corresponding to the entries 1 and $n+2$ as in the hint, resulting in two paths of length n . Let v_l be the vector consisting of entries 2 to $n+1$ of u_l and let P_n be the path with the corresponding vertices. Then for $l = 1, \dots, n$

$$\lambda_l = 2 \cos \left(l \cdot \frac{2\pi}{2n+2} \right) = 2 \cos \left(\frac{\pi l}{n+1} \right)$$

is an eigenvalue of P_n with eigenvector v_l .

3. Prove that if G is d -regular, then the multiplicity of the largest eigenvalue λ_1 equals the number of connected components of G .

For any component C , let v_C be a vector that has values 1 on this component and zero everywhere else. Then because G is regular this is clearly an eigenvector for λ_1 , and for different components we get linearly independent eigenvectors. So $m(\lambda_1)$ is at least the number of connected components. We'll be done if we can show that any other eigenvectors is a linear combination of these.

Let $x = (x_v)_{v \in V(G)}$ be an eigenvector of $\lambda_1 = d$, and take the largest nonzero entry x_u . Then

$$x_u = \frac{1}{d} \sum_{v \in N(u)} x_v.$$

So a sum with d terms, divided by d , gives something at least as large as all the terms. Then we must have $x_v = x_u$ for all $v \in N(u)$. Repeating this for neighbors of the neighbors, etc, gives that the eigenvector is constant on the whole connected component of u .

The same works for any component, so x is constant on each component (although not necessarily with the same value), which means that it is a linear combination of the v_C , which is what we wanted.

4. Prove that if G is connected, then the diameter of G is strictly less than its number of distinct eigenvalues.

The diameter D is the largest of the distances between pairs of vertices of the graph. (If G were not connected, this would not be defined or infinity, which we need that condition.) We can formulate this in terms of the adjacency matrix A : If (v_i, v_j) is a pair of vertices at distance D , then $(A^D)_{ij} \neq 0$ (there is a D -path between them), but $(A^k)_{ij} = 0$ for all $k < D$ (there is no shorter path). So D is the largest number for which a pair exists with this property.

Now to connect this with the number of distinct eigenvalues. If the d distinct eigenvalues of A are μ_1, \dots, μ_d , with multiplicities m_i , then the characteristic polynomial of A is

$$f(x) = \det(A - xI) = (-1)^n \prod_{i=1}^d (x - \mu_i)^{m_i}.$$

It is a well-known fact that A satisfies the polynomial

$$p(x) = \prod_{i=1}^d (x - \mu_i).$$

To see this, take a diagonalization $A = S^{-1}DS$, so $A^k = S^{-1}D^kS$, and

$$p(A) = S^{-1}p(D)S = 0,$$

since $p(D)$ is the matrix with diagonal entries $p(\mu_i)$, all of which are zero.

So A satisfies a polynomial of degree d , the number of distinct eigenvalues. Suppose $D \geq d$. Then from $0 = p(A) = A^d + q(A)$, with q of degree $< d$, we could get an equation

$$A^D = A^{D-d}A^d = -A^{D-d}q(A).$$

The right-hand side only involves powers A^k with $k < d$. But as observed above, A^D has some nonzero entry ij such that all lower powers have a zero there, which contradicts this equation. Done.
