Advanced Discrete Mathematics 2013 – Problem Set 1

You can hand in one of the star problems before 3pm on Monday March 4th.

1. Prove the following facts from linear algebra (over $\mathbb{R}$):

   (a) $\mathrm{rk}(A+B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$ for any two $m \times n$ matrices A, B;

   (b) $\mathrm{rk}(AB) \leq \min(\mathrm{rk}(A), \mathrm{rk}(B))$ for any $k \times l$ matrix A and $l \times m$ matrix B;

   (c) if an $n \times n$ matrix M is positive definite, then $\mathrm{rk}(M) = n$.

2. In the proof of Fisher's Inequality we used the fact that any matrix of the form $bJ + D$ is nonsingular, if $b \geq 0$ and D is a diagonal matrix with diagonal entries > 0. For example this one, if all $a_i > b \geq 0$:

$$\begin{pmatrix} a_1 & b & b & b \\ b & a_2 & b & b \\ b & b & a_3 & b \\ b & b & b & a_4 \end{pmatrix}.$$

   Prove this in 3 ways: using positive definiteness, the determinant, and the definition of linear independence.

3. Given a bound on the size of a certain kind of object, a *tight example* is such an object with size exactly equal to that bound.
   Give two families of tight examples for the bound for 1-intersecting set systems.
   In other words, give two constructions that for any $|X|$ give two tight examples for $|\mathcal{S}| \leq |X|$ (which are really different, ie not the same after relabelling). For convenience take $X = \{1, 2, \ldots, n\}$. Note that we are not requiring uniformity.

4. Given n points in $\mathbb{R}^2$ which are not collinear, prove that there are at least n lines that pass through at least two of these points.
   (Hint: Compare the fact that there is exactly one line through any two points with the condition of Fisher's Inequality.)

5. Show that there are at least 2^{n-4} different ways to decompose K_n into $n-1$ bicliques.
   Again they should be really different, ie not the same after relabelling.

6. Suppose we have sets $A_1, \ldots, A_{n+1} \subset X$ and $|X| = n$. Show that there are two disjoint sets I, J of indices such that
$$\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j.$$

7. * Given $|X| = n$ and distinct nonempty subsets $A_1, \ldots, A_n \subset X$. Show that there is an $x \in X$ such that the sets $A_i \backslash \{x\}$ are still distinct.

8. * The examples that you found in problem 3 are not uniform. Show that there exists a tight example for the bound $|\mathcal{S}| \leq |X|$ for *uniform* 1-intersecting set systems in the case $|X| = 7$.
   Show that in general this is only possible if $|X| = q^2 + q + 1$ for some integer $q > 0$.