Advanced Discrete Mathematics 2013 – Problem Set 1 – Solutions

1. *Prove the following facts from linear algebra (over $\mathbb{R}$):*

   (a) *$rk(A+B) \leq rk(A) + rk(B)$ for any two $m \times n$ matrices A, B;*
   We have to show that the dimension of the column space of $A+B$ is at most the sum of the dimensions of the column spaces of A and B.
   Let V be the span of the column vectors of A and B together. Its dimension is $\leq \text{rk}(A) + \text{rk}(B)$ because we can find a basis in the union of two bases of the columns spaces of A and B, and the sizes of these bases are $\text{rk}(A)$ and $\text{rk}(B)$.
   On the other hand, the column vectors of $A+B$ are in V because they are sums of column vectors of A and B. Hence the dimension of the span of the columns of $A+B$ is at most the dimension of V.

   (b) *$rk(AB) \leq \min(rk(A), rk(B))$ for any $k \times l$ matrix A and $l \times m$ matrix B;*
   Every column vector of AB is a linear combination of column vectors of A (with coefficients from a column vector of B), so the column space of AB is contained in the column space of A, hence $\text{rk}(AB) \leq \text{rk}(A)$. The same argument, but with row vectors, gives $\text{rk}(AB) \leq \text{rk}(B)$.

   (c) *if an $n \times n$ matrix M is positive definite, then $rk(M) = n$.*
   We defined a real symmetric matrix to be positive definite if $x^T M x > 0$ for all $0 \neq x \in \mathbb{R}^n$.
   Note: There are many very different but equivalent definitions. The one above is convenient for real symmetric matrices (which most of the matrices in this course will be), but it would not give the usual notion for non-real or non-symmetric matrices (which doesn't mean that it's *wrong*). In general, what one wants of positive definiteness is for the eigenvalues to be positive reals; for complex matrices, one would have to require $x \in \mathbb{C}^n$.
   If $\text{rk}(M) \neq n$, then the matrix is singular, so there is an $x \neq 0$ such that $Mx = 0$. Then $x^T M x = 0$.

2. *In the proof of Fisher's Inequality we used the fact that any matrix of the form $bJ + D$ is nonsingular, if $b \geq 0$ and D is a diagonal matrix with diagonal entries > 0. For example this one, if all $a_i > b \geq 0$:*

$$\begin{pmatrix} a_1 & b & b & b \\ b & a_2 & b & b \\ b & b & a_3 & b \\ b & b & b & a_4 \end{pmatrix}.$$

   Prove this in 3 ways: using positive definiteness, the determinant, and the definition of linear independence.

With positive definiteness:
This is the proof that we used in class. By 1c) we just have to show that $bJ + D$ is positive definite. So we calculate (with d_i the diagonal entries of D and x_i the entries of $x \neq 0$):

$$x^T(bJ+D)x = b\left(\sum x_i\right)^2 + \sum d_i x_i^2 > 0.$$

We have > 0 because it is a sum of squares with positive coefficients. Done.

With determinants:
We can compute the determinant of this kind of matrix without too much work, by using the fact that adding a multiple of one row or column to another leaves the determinant unchanged. This works best if we use the trick of adding a row of 1's. We'll demonstrate it on a 2×2 matrix, and it should be clear from there how to do it in general.

$$\det \begin{pmatrix} b+d_1 & b \\ b & b+d_2 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & b+d_1 & b \\ 0 & b & b+d_2 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 \\ -b & d_1 & 0 \\ -b & 0 & d_2 \end{pmatrix}$$
$$= \det \begin{pmatrix} 1+\frac{b}{d_1}+\frac{b}{d_2} & 1 & 1 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{pmatrix} = d_1 d_2 \left(1 + b\left(\frac{1}{d_1}+\frac{1}{d_2}\right)\right).$$

We want to show that the determant $\neq 0$. By assumption we have $d_1, d_2 \neq 0$, and then we need
$$\frac{1}{d_1}+\frac{1}{d_2} \neq -\frac{1}{b},$$
which follows by the assumptions that $d_i > 0$ and $b > 0$.
You can see that this way you would in general get
$$d_1 d_2 \cdots d_n \left(1 + b\left(\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_n}\right)\right).$$
So in general this matrix is nonsingular if and only if $d_1 \neq 0$, $d_2 \neq 0$, and
$$\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_n} \neq -\frac{1}{b}.$$

With linear independence:
Actually, I don't know a good way to do this using the definition of linear independence. In theory one could suppose $\sum \lambda_i v_i = 0$, with v_i being the column vectors, then view this as a linear system in the λ_i, and solve this with Gaussian elemination. But this is just a more complicated version of the determinant proof of above.
But note that when this matrix shows up in the proof of the generalized Fisher's inequality, we *can* prove this with linear independence, as in the first lecture. (To be honest, this is what I had in mind when I asked the question like this...) In that situation we know that the matrix is of the form $A^T A$ for some matrix A, so the column vectors v_i of this matrix A satisfy
$$\langle v_i, v_j \rangle = \begin{cases} b + d_i & \text{if } i = j \\ b & \text{if } i \neq j \end{cases}.$$
Then if $\sum \lambda_i v_i = 0$ we get
$$0 = \langle \sum \lambda_i, \sum \lambda_i \rangle = \sum_i (b+d_i)\lambda_i^2 + \sum_{i \neq j} b\lambda_i\lambda_j = \sum_i d_i \lambda_i^2 + \sum_{i,j} b\lambda_i\lambda_j$$
$$= \sum_i d_i \lambda_i^2 + b \cdot \left(\sum_i \lambda_i\right)^2.$$
By assumption the coefficients are positive, so all $\lambda_i = 0$.
This implies that A is nonsingular, so $\det(A) \neq 0$. Then also $\det(A^T A) = \det(A)^2 \neq 0$, so $A^T A$ is nonsingular.
Unfortunately, this approach only works if the matrix is of the form $A^T A$ with A nonsingular, which is actually equivalent to being positive definite.

3. *Given a bound on the size of a certain kind of object, a tight example is such an object with size exactly equal to that bound.*
   Give two families of tight examples for the bound for 1-intersecting set systems.
   You're meant to find them by just playing around. Here they are, as subsets of $[n] = \{1, 2, \ldots, n\}$:
   $$\{\{1\}, \{1,2\}, \{1,3\}, \ldots, \{1,n\}\};$$
   $$\{[n-1], \{1,n\}, \{2,n\}, \ldots, \{n-1,n\}\}.$$
   They clearly satisfy the conditions, and they are really different because one has a 1-elements sets and the other doesn't. Ok, for $n = 2$ they coincide, but we can ignore that.

4. *Given n points in $\mathbb{R}^2$ which are not collinear, prove that there are at least n lines that pass through at least two of these points.*
   This follows from the nonuniform Fisher's inequality. There we have subsets such that each pair has exactly one element in common, while in this question we have points such that each pair has exactly one line in common (two lines do have one point in common, but this need not be one of our n points).
   To make this more precise, suppose we have n points p_i and m lines l_j that pass through at least two of the points. Then we define a set system by taking $X = \{l_j\}$ with subsets $\mathcal{S} = \{S_i\}$, where
   $$S_i = \{l_j : p_i \in l_j\}.$$
   Then $S_i \cap S_{i'}$ is the unique line through p_i and $p_{i'}$, if $i \neq i'$, so this is a 1-intersecting set system. Hence $n = |\mathcal{S}| \leq |X| = m$, which is what we wanted to show.

5. *Show that there are at least 2^{n-4} different ways to decompose K_n into $n-1$ bicliques.*
   First we define the *star decomposition* of K_n as the following:
   $$K_n = K_{1(n-1)} + K_{1(n-2)} + \cdots + K_{12} + K_{11}.$$
   Note that there is only one star decomposition up to isomorphism, ie by relabelling you can identify any two. On the other hand, it is easy to distinguish a star decomposition from other ones with $n-1$ bicliques: If there is a K_{rs} with $r > 1$ or $s > 1$, then it's not a star decomposition. We will use this below to ensure that we really get different decompositions.

   We will use induction on n to prove something slightly stronger: There are 2^{n-4} different ways to decompose K_n *not counting the star decomposition.* For $n = 4$ it is true since there is clearly a way to decompose K_4 into 3 bicliques which is not a star decomposition:
   $$K_4 = K_{22} + K_{11} + K_{11}.$$
   Now assume it is true for $n-1$ and smaller, and consider K_n.
   For each $1 \leq i \leq n-4$, we can decompose it as
   $$K_n = K_{i(n-i)} + K_{n-i} + K_i,$$
   and then decompose the K_{n-i} and K_i into bicliques. For different i these decompositions are really different, because each $K_{i(n-i)}$ does not occur in any of the others.
   By induction there are 2^{n-4-i} different ways to decompose K_{n-i} into bicliques without the star decomposition. We use each of these. For the other part, K_i, we will just use a single decomposition each time, namely the star decomposition (for $i = 1$ we just

take the single vertex and count it as 1). For a fixed i, the resulting decompositions are then really different, because on one side of $K_{i(n-i)}$ there will be a star decomposition, while on the other side there won't be.
Now the total number of non-star decompositions we have is

$$\sum_{i=1}^{n-4} 2^{n-4-i} = 2^{n-4} - 1.$$

We need one more to get the induction claim. Take $i = n-5$, so $K_n = K_{(n-5)5} + K_5 + K_{n-5}$, and use a non-star decomposition of K_5 and a star decomposition of K_{n-5}.

6. *Suppose we have sets $A_1, \ldots, A_{n+1} \subset X$ and $|X| = n$. Show that there are two disjoint sets I, J of indices such that*
$$\bigcup_{i\in I} A_i = \bigcup_{j \in J} A_j.$$
Consider the characteristic vectors a_i of the sets A_i. These are $n+1$ vectors in $\mathbb{R}^n$, so they must be linearly dependent. That means there is a linear relation $\sum_{i=1}^{n+1} c_i a_i = 0$ with not all $c_i = 0$. Define
$$I = \{i \in [n+1] : c_i > 0\}, \quad J = \{j \in [n+1] : c_j < 0\},$$
so that
$$\sum_{i\in I} c_i a_i = \sum_{j\in J} (-c_j) a_j.$$
We claim that this I and J are as in the question.
Consider an $x \notin \cup_{i\in I} A_i$. Then the entry corresponding to x is 0 for all the a_i with $i \in I$. Then from the equation above that entry must also be 0 in $\sum_{j\in J}(-c_j)a_j$. Since all the coefficients $-c_j$ in that sum are positive, this implies that each a_j with $j \in J$ has 0 in the entry corresponding to x, which means that $x \notin \cup_{j\in J} A_j$.
On the other hand, if $x \in \cup_{i\in I} A_i$, then in some a_i with $i \in I$ its entry is nonzero. Since all the coefficients on the left of the equation above are positive, this entry is also nonzero in $\sum_{i\in I} c_i a_i$, which implies that it is nonzero in $\sum_{j\in J}(-c_j)a_j$. That finally means that $x \in \cup_{j\in J} A_j$.
Note that we had to make sure that all the coefficients were positive, otherwise there could have been cancellation, and the arguments would not have worked.