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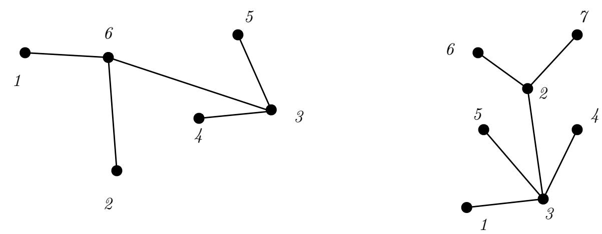
Exercises Week 2

1. Using Prüfer's algorithm find the sequences corresponding to the the following two trees:



2. Construct the trees corresponding to the following Prüfer codes:

   (a) 76543

   (b) 333445.

   Definition 1. *A graph G is* bipartite *if there is a partition of the vertices V into two subsets V_1 and V_2 such that every edge of G has one endpoint in V_1 and one in V_2.*

3. A graph is bipartite if and only if it has no odd cycles.

   Definition 2. *A graph is* complete *if it has an edge between every pair of vertices. We use K_n to denote the complete graph with n vertices. We say that K_3 is a* triangle.

   Definition 3. *For a graph G we say that a subset U of its vertices is* independent *if there is no edge with both of its endpoints in U .*

4. Let G be a graph containing no triangles. Show that there is an independent set of size $\sqrt{|V(G)|} - 1$.

   Definition 4. *For a graph G, we say that a sequence of vertices $v_0, ..., v_k$ is a walk if $v_i v_{i+1} \in E(G)$ for all $i \in \{0, 1..., k-1\}$ (note that a vertex may appear more than once in a walk). A* closed walk *is a walk that starts and ends at the same vertex. We say that a graph is* Eulerian *if there is a closed walk that passes through every edge exactly once and it is connected.*

5. Show that a graph is Eulerian if and only if it is connected and every vertex has an even degree.
   Hint: In order to show that a connected graph where every vertex has even degree is Eulerian, show that a maximal walk that uses every edge at most once necessarily starts and ends at the same vertex. Then show that if this does not cover all the edges, then it can be prolonged, contradicting the fact that it is maximal.

6. We say that a graph has an Eulerian walk if there is a walk that passes through each edge exactly once. Show that a connected graph has an Eulerian walk if and only if there are at most two odd degree vertices.

> Bonus Problem: *Prove that in a graph any matching without an augmenting path is of maximal cardinality. (Note that G does not need to be bipartite, and furthermore Hall's and Konig's theorem are only for bipartite graphs!)*

- The assignment is due on Thursday, March 5 at the exercise session
- Submit a solution to the bonus problem *only*.