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Multi-Level Risk Aggregation

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Abstract

In this paper we compare the current Solvency II standard and a genuine bottom-up approach to risk aggregation. This is understood to be essential for developing a deeper insight into the possible differences between the diversification assumptions between the standard approach and internal models.

1 Introduction

There are various approaches to model diversification benefits using linear correlation at solo level. In the current Solvency II standard two-level approach, there is a base correlation matrix within each risk class (market, life, non-life, health, default) and a top level correlation matrix between these risk classes. Internal models tend to follow a genuine bottom-up approach which uses a correlation matrix that combines all risk types. See e.g. the recent CEIOPS documents [1, 2] and the CRO Forum QIS3 Benchmarking study [6].

In this paper we compare the two approaches. In particular, we discuss the interplay between the top level correlation between risk classes and the base correlation matrix between the risk types. This is understood to be essential for developing a deeper insight into the possible differences between the diversification assumptions between the standard approach and internal models.

In Section 2, we show that in general only correlation parameters set at the base level lead to unequivocally comparable solvency capital requirements across the industry (Theorem 2.2). This also supports the findings in the technical paper [7] of the Groupe Consultatif. In Section 3, we then consider portfolio dependent base correlations. These implied correlations are not unique in general. We provide at least three possible specifications and give sufficient conditions such that they actually qualify as positive semi-definite correlation matrices.

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We then show that there exists a unique minimal base correlation matrix (Theorem 3.2). This distinguished correlation matrix may serve as a benchmark for comparison of standard and internal correlation specifications. In Section 4, we compute the three base correlation matrices for a life and non-life insurance portfolio which reflect the EEA average from the QIS3 Benchmarking Study of the CRO Forum [6]. These results may be used as benchmark for a possible standard specification of base correlations between market, life and non-life risk types. Section 5 concludes.

2 Constant Base Correlation

For the sake of illustration, we consider two risk classes composed of m and n risk types with stand alone solvency capital requirements (i.e. 99.5%-quantiles) given by the vectors

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}_+^m \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}_+^n.$$

Here and subsequently, with a vector we mean a column vector. It is straightforward to extend the following to more than two risk classes.

The current Solvency II standard model is based on a two-level correlation aggregation. First, some $m \times m$ and $n \times n$ base correlation matrices A for x and B for y yield the solvency capital requirements per risk class

$$X = \sqrt{x^T \cdot A \cdot x} \quad \text{and} \quad Y = \sqrt{y^T \cdot B \cdot y}, \tag{1}$$

respectively. Second, some top level correlation factor R between X and Y yields the total solvency capital requirement

$$SCR = \sqrt{X^2 + 2RXY + Y^2}. \tag{2}$$

The standard model provides base and top level correlation parameters A, B and R. Input from the insurance company are the portfolio specific data x, y.

A genuine bottom-up model, in contrast, uses a full $(m+n) \times (m+n)$ base correlation matrix

$$M = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \tag{3}$$

that aggregates all risk types, across risk classes, in one go:

$$SCR = \sqrt{(x^T, y^T) \cdot M \cdot \begin{pmatrix} x \\ y \end{pmatrix}}. \tag{4}$$

Equalling (2) and (4) implies

$$x^T \cdot C \cdot y = R\sqrt{x^T \cdot A \cdot x}\sqrt{y^T \cdot B \cdot y}. \tag{5}$$

It is understood that the full base correlation matrix M is fundamental part of the risk model. It reflects the underlying nature of the risks, which is generic and company independent. Company specific, in contrast, is the individual exposure to the risks. Thus, M should simultaneously apply to all companies. In that sense, M should not depend on the company specific portfolio x, y, while the implied top level correlation

$$R = R(x, y) = \frac{x^T \cdot C \cdot y}{\sqrt{x^T \cdot A \cdot x}\sqrt{y^T \cdot B \cdot y}}$$

then does. Conversely:

Definition 2.1. *For A, B, R given, we call $C(x, y)$ a* **base correlation matrix** *for x, y if M in* (3) *is a correlation matrix (i.e. positive semi-definite) and* (5) *is satisfied.*

Now suppose the standard model (1)–(2) specified by A, B, R is applied by N companies with portfolio data $x(i), y(i)$, $i = 1, \ldots, N$. We then shall say that the resulting solvency capital requirements $SCR(i)$ are **unequivocally comparable** if there exists a common base correlation matrix $C(x(i), y(i)) \equiv C$ for all $i = 1, \ldots, N$. In other words, the solvency capital requirements are unequivocally comparable if they are based on a common underlying risk model.

Checking for unequivocal comparability is an **inverse problem**, which is difficult to solve in general. Indeed, (5) can be seen as linear system of N equations for the mn-vector (C_{ij}):

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i(k) y_j(k) C_{ij} = z(k), \quad k = 1, \ldots, N, \tag{6}$$

with $z(k) := R\sqrt{x^T(k) \cdot A \cdot x(k)}\sqrt{y^T(k) \cdot B \cdot y(k)}$. Since the set of solutions C to (6) equals the (possibly empty) preimage of the N-vector $z(k)$ for the $N \times (mn)$-matrix $(x_i(k)y_j(k))$, a necessary and sufficient condition for the existence of a solution C of (6) is that the N-vector $(z(k))$ lies in the image of the $N \times (mn)$-matrix $(x_i(k)y_j(k))$. For a generic vector $z(k)$ in $\mathbb{R}_+^N$ this essentially requires that $N \leq mn$, see e.g. [5, Paragraph 0.2.3]. But the number of tested companies will typically be greater than mn (there are currently six risk types in market and seven in life, that is $mn = 42$). Moreover, even if a solution C of (6) exists, it yet has to satisfy that M in (3) is positive semi-definite.

The following example shows that a solution may not exist if $N > mn$: let $m = 2$, $n = 1$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = (1)$, $R = 0.4$, and $N = 3$. The portfolio data are $x(1) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $x(2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $x(3) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $y(1) = y(2) = y(3) = 1$ (y actually cancels out in (5) and (6), respectively). Then $x(1), x(2)$ already uniquely determine the solution $C = \begin{pmatrix} 0,26 \\ 0,30 \end{pmatrix}$ of (6). But a straightforward inspection shows that (5) does not hold for $x(3)$.

The next result shows that the above inverse problem is in fact generically ill-posed. Indeed, if we assume infinitely many companies with a continuum of portfolio data $(x,y) \in \mathbb{R}_+^{m+n}$, then a common base correlation matrix $C(x,y) \equiv C$ exists for all $(x,y) \in \mathbb{R}_+^{m+n}$ if only if the risk types are either uncorrelated or fully correlated. Denote by $J_{m\times n}$ the $m \times n$-matrix with all entries equal 1.

Theorem 2.2. *Suppose A, B, R are given as above. Then there exists a common base correlation matrix $C(x,y) \equiv C$ for all $(x,y) \in \mathbb{R}_+^{m+n}$ if and only if $C = RJ_{m\times n}$ and either $R = 0$ or $A = J_{m\times m}$ and $B = J_{n\times n}$.*

Proof. Sufficiency of the statement is clear. For necessity, we insert $x = e_i$ and $y = e_k$ (the standard basis vectors in $\mathbb{R}^m$ or $\mathbb{R}^n$, respectively) in (5) and obtain $C = RJ_{m\times n}$. If $R = 0$ we are done. Otherwise, we divide (5) by R and square on both sides to obtain

$$\sum_{i,j=1}^{m} \sum_{k,l=1}^{n} x_i x_j y_k y_l = \sum_{i,j=1}^{m} \sum_{k,l=1}^{n} A_{ij} B_{kl} x_i x_j y_k y_l.$$

Matching coefficients yields $A_{ij}B_{kl} = 1$ and, since $A_{ii} = B_{kk} = 1$, thus the claim. □

Based on Theorem 2.2, we may state that in general only correlation parameters set at the base level lead to unequivocally comparable solvency capital requirements across the industry.

3 Minimal Base Correlation

Theorem 2.2 showed that it is generically impossible to find a common base correlation matrix $C(x,y) \equiv C$ for all $(x,y) \in \mathbb{R}_+^{m+n}$.

In this section, we relax this assumption and find base correlation matrices $C(x,y)$ which may depend on the given portfolio data $(x,y) \in \mathbb{R}_+^{m+n}$. Such solutions exist, but are not unique, in general. The next lemma provides some examples. For two vectors $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$ we write $u \cdot v^T$ for the $m \times n$-matrix $(u_i v_j)$. We denote by $\|D\| = \sqrt{\mathrm{tr}(D \cdot D^T)}$ the Euclidian norm of a matrix D. Hence, in particular, $\|u\| = \sqrt{u_1^2 + \cdots, u_m^2}$ for an m-vector u. It follows by inspection that the following matrices satisfy (5):

$$C(x,y) = R \frac{A \cdot x}{\sqrt{x^T \cdot A \cdot x}} \cdot \frac{y^T \cdot B}{\sqrt{y^T \cdot B \cdot y}}, \tag{7}$$

$$C(x,y) = R \frac{\sqrt{x^T \cdot A \cdot x}\sqrt{y^T \cdot B \cdot y}}{\|x\|^2 \|y\|^2} x \cdot y^T, \tag{8}$$

$$C(x,y) = R \frac{\sqrt{x^T \cdot A \cdot x}\sqrt{y^T \cdot B \cdot y}}{x^T \cdot J_{m\times n} \cdot y} J_{m\times n}. \tag{9}$$

We next provide sufficient conditions on A and B such that examples (7)–(9) qualify as base correlation matrix.

Lemma 3.1. (i) (7) *is a base correlation matrix for* x, y.

(ii) *If there exists* $p, q \geq 0$ *with* $pq = R^2$ *such that both matrices*

$$A - p\frac{x^T \cdot A \cdot x}{\|x\|^4} x \cdot x^T \quad \text{and} \quad B - q\frac{y^T \cdot B \cdot y}{\|y\|^4} y \cdot y^T \tag{10}$$

are positive semi-definite, then (8) *is a base correlation matrix for* x, y.

(iii) *If there exists* $p, q \geq 0$ *with* $pq = R^2$ *such that both matrices*

$$A - p\frac{x^T \cdot A \cdot x}{x^T \cdot J_{m\times m} \cdot x} J_{m\times m} \quad \text{and} \quad B - q\frac{y^T \cdot B \cdot y}{y^T \cdot J_{n\times n} \cdot y} J_{n\times n} \tag{11}$$

are positive semi-definite, then (9) *is a base correlation matrix for* x, y.

Proof. It remains to show that M in (3) is positive semi-definite, that is,

$$u^T \cdot A \cdot u + 2u^T \cdot C(x, y) \cdot v + v^T \cdot B \cdot v \geq 0, \quad \forall (u, v) \in \mathbb{R}^{m+n}.$$

This is equivalent to

$$(u^T \cdot A \cdot u)(v^T \cdot B \cdot v) \geq (u^T \cdot C(x, y) \cdot v)^2, \quad \forall (u, v) \in \mathbb{R}^{m+n}. \tag{12}$$

For (7), property (12) follows by the Cauchy–Schwarz inequality $(u^T \cdot A \cdot x)^2 \leq (u^T \cdot A \cdot u)(x^T \cdot A \cdot x)$ and analogously for B. This proves (i).

For (8), property (12) holds if and only if

$$(u^T \cdot A \cdot u)(v^T \cdot B \cdot v) \geq R^2 \frac{x^T \cdot A \cdot x}{\|x\|^2} \frac{y^T \cdot B \cdot y}{\|y\|^2} \frac{u^T \cdot x \cdot x^T \cdot u}{\|x\|^2} \frac{v^T \cdot y \cdot y^T \cdot v}{\|y\|^2}$$

for all $(u, v) \in \mathbb{R}^{m+n}$. This proves (ii).

Part (iii) follows similarly. □

We now show that (8) is distinguished among all base correlation matrices.

Theorem 3.2. *Suppose* $C^* = C(x, y)$ *in* (8) *is a base correlation. Then it is minimal in the following sense:*

$$\|C^*\| = \min \{\|C\| \mid C \textit{ is base correlation matrix for } x, y\}.$$

Proof. We consider the scalar product $\langle C, D\rangle = \operatorname{tr}(C \cdot D^T)$ on $\mathbb{R}^{m\times n}$. Then $\|C\| = \sqrt{\langle C, C\rangle}$, and the left hand side of (5) is just the scalar product:

$$x^T \cdot C \cdot y = \langle C, x \cdot y^T\rangle.$$

Hence every C that satisfies (5) can uniquely be decomposed into

$$C = R\frac{\sqrt{x^T \cdot A \cdot x}\sqrt{y^T \cdot B \cdot y}}{\|x\|^2\|y\|^2} \times (x \cdot y^T + N)$$

where N is orthogonal to $x \cdot y^T$, i.e. $\langle x \cdot y^T, N\rangle = 0$. Hence

$$\|C\|^2 = \|C^*\|^2 + R^2\frac{(x^T \cdot A \cdot x)(y^T \cdot B \cdot y)}{\|x\|^4\|y\|^4}\|N\|^2 \geq \|C^*\|^2$$

with equality if and only if $N = 0$. □

By the very definition, for given A, B, R and x, y, every base correlation matrix for x, y yields the same total solvency capital requirement SCR. Hence there is no way in distinguishing a base correlation matrix for x, y by its diversification effect for the portfolio (x, y). Nonetheless, from Theorem 3.2 we may infer the following universal minimality property of $C^* = C(x, y)$ in (8). Suppose C is any other base correlation matrix for x, y. Then Theorem 3.2, combined with the Cauchy–Schwarz inequality (see e.g. [5, Paragraph 0.6.3]), says that the maximal weighted sum of the entries of C^* is strictly smaller than the respective maximal sum for C in the sense that

$$\sup_{\|D\|=1} \sum_{i=1}^{m} \sum_{j=1}^{n} D_{ij} C^*_{ij} = \|C^*\| < \|C\| = \sup_{\|D\|=1} \sum_{i=1}^{m} \sum_{j=1}^{n} D_{ij} C_{ij}.$$

In this sense, (8) allocates the prescribed top level correlation R among the base risk types in a uniquely minimal way, as illustrated in the next section.

In terms of diversification effects, this can be expressed as follows. Suppose A, B and some base correlation matrix $C = C(x, y)$ calibrated to the portfolio (x, y) are going to be used as benchmark risk model for other portfolios. Moreover, suppose we measure the diversification effect for any portfolio $(\xi, \eta) \in \mathbb{R}_+^{m+n}$ as difference between the squared total solvency capital requirement from the squared solvency capital requirement with zero top level correlation:

$$\Delta(\xi, \eta, C) = \xi^\top A \xi + 2\xi^\top C \eta + \eta^\top B \eta - \left(\xi^\top A \xi + \eta^\top B \eta\right) = 2\xi^\top C \eta.$$

That is, the less $\Delta(\xi, \eta, C)$, the higher the diversification effect. It then follows as above that

$$\sup_{\|\xi\|\|\eta\| \le \|x\|\|y\|} \Delta(\xi, \eta, C^*) = 2x^\top C^* y \le \sup_{\|\xi\|\|\eta\| \le \|x\|\|y\|} \Delta(\xi, \eta, C).$$

In words, the lowest diversification effect among all portfolios $(\xi, \eta) \in \mathbb{R}_+^{m+n}$ with $\|\xi\|\|\eta\| \le \|x\|\|y\|$ for C^* is higher than the respective lowest diversification effect for any other base correlation matrix $C = C(x, y)$.

4 Application to QIS3 Data

Figures 1 and 2 in the appendix show the EEA-average solvency capital requirements per risk type for a life and non-life insurer, respectively, taken from the QIS3 Benchmarking Study[1] of the CRO Forum [6].

Figures 3–6 show the top and base level correlation matrices according to the QIS3 standard model [1].

[1]These figures are derived from the proportion splits of QIS3 capital charges as shown on pages 29, 31, 55 and 39, 41, 43 in the document [6]. The capital requirements are thus normalized such that the Basic SCR results in 100×100. The risk class "default" is negligible, both for life and non-life insurers, and therefore is omitted.

One then checks numerically by computing the eigenvalues that the two matrices in (10) (for $p = 0.1$ and $q = 0.625$) and in (11) (for $p = q = 0.25$) are positive semi-definite, both for the life and non-life portfolio. By Lemma 3.1 it follows that all examples (7)–(9) are base correlations for the given capital requirements in Figures 1 and 2.

The resulting base correlations between market and life and non-life risk types for the life and non-life insurer, respectively, are shown in Figures 7–9 and 10–12. Cells with correlations greater than 0.1 are indicated.

It becomes obvious that the minimal base correlation matrix (8) assigns less correlation to risk types than the other two examples (7) and (9).

5 Conclusion

In this paper, we rigorously demonstrated the fact that only correlation parameters set at the base level lead to unequivocally comparable solvency capital requirements across the industry.

Relaxing the assumptions, we then found portfolio dependent base correlation matrices that correspond to a prescribed top level correlation. Narrowing further the choice we arrived at a unique minimal solution, which we then explicitly computed for QIS3 data from [6]. I suggest that further empirical comparison of standard and internal correlation specifications is carried out with this minimal solution as a benchmark. However, I also stress the fact that Value-at-Risk and correlation aggregation does not appropriately capture tails and tail dependence of risks in the insurance business. In that regard, I encourage the additional use of risk and dependence modeling beyond correlation such as indicated in e.g. [4, 3].

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A Results

Mkt	int	1536
	eq	2624
	prop	512
	sp	1408
	conc	64
	fx	256
Life	mort	140
	long	1190
	dis	245
	lapse	700
	exp	385
	rev	0
	CAT	840

Figure 1: EEA-average solvency capital requirements per risk type for a life insurer. Source: [6].

Mkt	int	572
	eq	2508
	prop	396
	sp	264
	conc	572
	fx	132
NL	pr	4187
	CAT	1113

Figure 2: EEA-average solvency capital requirements per risk type for a non-life insurer. Source: [6].

BSCR	mkt	def	life	health	nl
mkt	1	0,25	0,25	0,25	0,25
def	0,25	1	0,25	0,25	0,5
life	0,25	0,25	1	0,25	0
health	0,25	0,25	0,25	1	0
nl	0,25	0,5	0	0	1

Figure 3: Top level correlation matrix between risk classes. Source: [1].

Mkt	int	eq	prop	sp	conc	fx
int	1	0	0,5	0,25	0	0,25
eq	0	1	0,75	0,25	0	0,25
prop	0,5	0,75	1	0,25	0	0,25
sp	0,25	0,25	0,25	1	0	0,25
conc	0	0	0	0	1	0
fx	0,25	0,25	0,25	0,25	0	1

Figure 4: Base level correlation matrix between market risk types. Source: [1].

Life	mort	long	dis	lapse	exp	rev	CAT
mort	1	0	0,5	0	0,25	0	0
long	0	1	0	0,25	0,25	0,25	0
dis	0,5	0	1	0	0,5	0	0
lapse	0	0,25	0	1	0,5	0	0
exp	0,25	0,25	0,5	0,5	1	0,25	0
rev	0	0,25	0	0	0,25	1	0
CAT	0	0	0	0	0	0	1

Figure 5: Base level correlation matrix between life risk types. Source: [1].

NL	pr	CAT
pr	1	0
CAT	0	1

Figure 6: Base level correlation matrix between non-life risk types. Source: [1].

	mort	long	dis	lapse	exp	rev	CAT
int	0,02	0,10	0,03	0,08	0,08	0,03	0,05
eq	0,04	0,15	0,05	0,12	0,12	0,04	0,08
prop	0,04	0,16	0,05	0,13	0,13	0,04	0,09
sp	0,03	0,11	0,04	0,09	0,09	0,03	0,07
conc	0,00	0,00	0,00	0,00	0,00	0,00	0,00
fx	0,02	0,08	0,03	0,06	0,06	0,02	0,04

Figure 7: Base level correlation matrix (7) between market and life risk types.

	mort	long	dis	lapse	exp	rev	CAT
int	0,01	0,12	0,02	0,07	0,04	0,00	0,08
eq	0,02	0,20	0,04	0,12	0,07	0,00	0,14
prop	0,00	0,04	0,01	0,02	0,01	0,00	0,03
sp	0,01	0,11	0,02	0,06	0,04	0,00	0,08
conc	0,00	0,00	0,00	0,00	0,00	0,00	0,00
fx	0,00	0,02	0,00	0,01	0,01	0,00	0,01

Figure 8: Minimal base level correlation matrix (8) between market and life risk types.

	mort	long	dis	lapse	exp	rev	CAT
int	0,09	0,09	0,09	0,09	0,09	0,09	0,09
eq	0,09	0,09	0,09	0,09	0,09	0,09	0,09
prop	0,09	0,09	0,09	0,09	0,09	0,09	0,09
sp	0,09	0,09	0,09	0,09	0,09	0,09	0,09
conc	0,09	0,09	0,09	0,09	0,09	0,09	0,09
fx	0,09	0,09	0,09	0,09	0,09	0,09	0,09

Figure 9: Uniform base level correlation matrix (8) between market and life risk types.

	pr	CAT
int	0,07	0,02
eq	0,23	0,06
prop	0,21	0,06
sp	0,09	0,02
conc	0,04	0,01
fx	0,08	0,02

Figure 10: Base level correlation matrix (7) between market and non-life risk types.

	pr	CAT
int	0,06	0,02
eq	0,26	0,07
prop	0,04	0,01
sp	0,03	0,01
conc	0,06	0,02
fx	0,01	0,00

Figure 11: Minimal base level correlation matrix (8) between market and non-life risk types.

	pr	CAT
int	0,14	0,14
eq	0,14	0,14
prop	0,14	0,14
sp	0,14	0,14
conc	0,14	0,14
fx	0,14	0,14

Figure 12: Uniform base level correlation matrix (8) between market and non-life risk types.