Optimal Capital and Risk Allocations for Law- and Cash-Invariant Convex Functions

Damir Filipović and Gregor Svindland

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Abstract

In this paper we provide the complete solution to the existence and characterization problem of optimal capital and risk allocations for not necessarily monotone, law-invariant convex risk measures on the model space L^p, for any $p \in [1, \infty]$. Our main result says that the capital and risk allocation problem always admits a solution via contracts whose payoffs are defined as increasing Lipschitz continuous functions of the aggregate risk.

Key words: exact convolutions, law-invariant risk measures, optimal capital and risk allocations.

1   Introduction

The problem of optimal capital and risk allocation among economic agents, or business units, has played a predominant role in the respective academic and industrial research areas for decades. The introduction of coherent and convex risk measures by Artzner et al. [3], Föllmer and Schied [19], and Frittelli and Rosazza-Gianin [20], respectively, has drawn the attention to study this problem using a new kind of approach (see Barrieu and El Karoui [5], Jouini et al. [21], Filipović and Kupper [14], Burgert and Rüschendorf [7], Acciaio [1], Ludkovski and Rüschendorf [24]). For some overview of the vast related finance literature we refer to Dana and Scarsini [11], Burgert and Rüschendorf [7] and the references therein.

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In this paper we provide the complete solution to the existence and characterization problem of optimal capital and risk allocations for not necessarily monotone, law-invariant convex risk measures on the model space L^p, for any $p \in [1, \infty]$. That is, we consider n agents, or business units, with initial endowments $X_1, \ldots, X_n \in L^p$, who assess the riskiness of their positions by means of some not necessarily monotone, law-invariant convex risk measures ρ_i on L^p. In order to minimize total and individual risk, the agents redistribute the aggregate endowment $X = X_1 + \cdots + X_n$ among themselves. An optimal capital and risk allocation $Y_1, \ldots, Y_n$ satisfies $Y_1 + \cdots + Y_n = X$ and

$$\rho_1(Y_1) + \cdots + \rho_n(Y_n) = \inf_{\sum_{i=1}^{n} Z_i = X} \left(\rho_1(Z_1) + \cdots + \rho_n(Z_n) \right). \qquad (1.1)$$

As is often the case in practice, this redistribution procedure may be subject to frictions (e.g. limited fungibility of capital, see [15, 16]) in the sense that not every allocation of X is admissible. This can be formalized by restricting the risk measures ρ_i accordingly, as proposed in [14], see also section 7 below. The restricted ρ_i are typically not monotone, even though the original ρ_i may be so. But this goes well with our framework, since we do not require monotonicity of ρ_i right from the start. Examples for ρ_i are mean-risk type risk measures, such as mean-variance, which obviously are convex law- and cash-invariant, but not monotone on L^p.

Our main result says that there *always* exists a solution to (1.1) in L^p of the form $Y_i = f_i(X)$, for some increasing Lipschitz continuous functions $f_i : \mathbb{R} \to \mathbb{R}$ with $\sum_{i=1}^{n} f_i = \mathrm{Id}_{\mathbb{R}}$. In other words, the capital and risk allocation problem (1.1) always admits a solution via contracts whose payoffs are defined as (increasing Lipschitz continuous) functions of the aggregate risk X. This extremely useful fact is often assumed in economic contract theory. We now set this prevalent economic assumption on a sound mathematical basis.

As regards the uniqueness of the optimal allocation, one has always the freedom to rebalance the cash (section 3). We will show that some kind of strict convexity of ρ_1 is a sufficient condition for uniqueness of the optimal allocation up to rebalancing the cash. However, we also provide an example which illustrates that this condition is not necessary.

The existence proof is constructive. Following along the lines of Landsberger and Meilijson [25], we approximate the optimal allocation by simple random variables. At each level of approximation, the respective approximate solution is comonotone and optimal with respect to the approximate aggregate endowment. This allows, in principle, to compute the approximate optimal capital and risk allocation at any given level of accuracy. A useful fact, that will be explored elsewhere.

The article of Jouini et al. [21] has been most influential for this paper. Indeed, Jouini et al. [21] provide existence of optimal allocations for *monotone* law-invariant convex risk measures ρ_i *on L^∞*. Our motivation was to understand and extend their results beyond L^∞, which from an applied perspective is a very limited model space (e.g. L^∞ does not contain normal distributed random variables). Moreover, in view of the predominant use of mean-variance risk

preferences in the literature and also the framework in [14], it was necessary to abandon the monotonicity assumption. Acciaio [1] provides further examples of this kind.

The remainder of the paper is as follows. In section 2, we state our main existence result (theorem 2.5) and illustrate it with various examples. In section 3, we elaborate on the uniqueness of optimal allocations. In section 4, we reduce the n-agent problem on L^p to the case $n = 2$ and $p = 1$. Section 5 contains the core of the proof of theorem 2.5. This is a result of Landsberger and Meilijson [25], which however they only proved for simple random variables. We thus provide a full and comprehensive proof. In section 6, we accomplish the proof of theorem 2.5. Section 7 discusses the above mentioned optimal allocation problem under constraints. Sections A and B contain lemmas that are needed for the proof of our main results.

2   Existence of Optimal Allocations

Throughout this paper $(\Omega, \mathcal{F}, \mathbb{P})$ denotes an atom-less standard probability space. All equalities and inequalities between random variables are understood in the $\mathbb{P}$-almost sure (a.s.) sense. If not specified otherwise, in the sequel, we let $p \in [1, \infty]$, and write $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$ and $\|\cdot\|_p = \|\cdot\|_{L^p}$. The topological dual space of L^p is denoted by $(L^p)^*$. It is well known that $(L^p)^* = L^q$ with $q = \frac{p}{p-1}$ for $p < \infty$, and that $(L^\infty)^* \supset L^1$ can be identified with ba, the space of all bounded finitely additive signed measures on $(\Omega, \mathcal{F})$ which vanish on $\mathbb{P}$-null sets. With some facilitating abuse of notation, we shall write $(X, Z) \mapsto E[XZ]$ for the dual pairing on $(L^p, (L^p)^*)$ also for the case $p = \infty$.

We suppose the reader is familiar with standard terminology and basic duality theory for convex functions as outlined in [26] or [13]. We call a function $F : L^p \to [-\infty, \infty]$

(i) *convex* if $F(\lambda X + (1-\lambda)Y) \leq \lambda F(X) + (1-\lambda)F(Y)$ for all $\lambda \in [0, 1]$ (here $\infty - \infty := \infty$),

(ii) *proper* if $F > -\infty$ and the *(effective) domain* $\operatorname{dom} F := \{F < \infty\} \neq \emptyset$,

(iii) *cash-invariant* if $F(0) \in \mathbb{R}$ and $F(X + m) = F(X) - m$ for all $m \in \mathbb{R}$,

(iv) *monotone* if $X \geq Y$ implies $F(X) \leq F(Y)$,

(v) *positively homogeneous* if $F(tX) = tF(X)$ for all $t \geq 0$,

(vi) *law-invariant* if $F(X) = F(Y)$ for all identically distributed $X \sim Y$.

The *conjugate function*

$$F^*(Z) = \sup_{X \in L^p} \left(E[ZX] - F(X) \right)$$

of F is *lower semi-continuous (l.s.c.)* and convex on $(L^p)^*$. The Fenchel–Moreau theorem (proposition 4.1 in [13]) states that $F^{**} = F$ if F is l.s.c. proper convex. The *indicator function* of a set $\mathcal{C}$ is defined as

$$\delta(X \mid \mathcal{C}) := \begin{cases} 0, & X \in \mathcal{C} \\ +\infty, & X \notin \mathcal{C}. \end{cases}$$

Definition 2.1. *A* convex risk measure *on L^p is a monotone convex cash-invariant function $L^p \to (-\infty, \infty]$. A* coherent risk measure *is a convex risk measure which in addition is positively homogeneous.*

Convex risk measures on L^p and more general model spaces have been studied, among others, in [6, 8, 18, 23]. Let $\rho : L^p \to (-\infty, \infty]$ be a convex cash-invariant function. The set of all acceptable positions with respect to ρ is $\mathcal{A}_\rho := \{X \in L^p \mid \rho(X) \le 0\}$. Note that, by cash-invariance, ρ is l.s.c. if and only if its acceptance set $\mathcal{A}_\rho$ is closed.

Prominent examples of convex risk measures on L^p are

(i) Average Value at Risk at level $\alpha \in [0, 1]$, defined as

$$\text{AVaR}_\alpha(X) := \begin{cases} -\frac{1}{\alpha} \int_0^\alpha q_X(s) \, ds, & \alpha > 0, \\ -\text{essinf}(X), & \alpha = 0 \end{cases} \tag{2.2}$$

where q_X denotes the left-continuous quantile function of X, in particular, $\text{AVaR}_1(X) = E[-X]$,

(ii) the entropic risk measure with parameter $\beta > 0$, defined as

$$\text{Entr}_\beta(X) = \frac{1}{\beta} \log E[e^{-\beta X}]. \tag{2.3}$$

It is well known, and can be found in e.g. [18], that AVaR_α is continuous on L^p for $\alpha > 0$, and $-\text{essinf}$ and Entr_β are l.s.c. on L^p and continuous on L^∞.

We now formalize the above capital and risk allocation problem. Let $n \ge 2$ and $F_1, \dots, F_n : L^p \to (-\infty, \infty]$ some proper convex functions. Their (infimal) *convolution* at $X \in L^p$ is defined as

$$\square_{i=1}^n F_i(X) = F_1 \square \dots \square F_n(X) := \inf_{\substack{X_1, \dots, X_n \in L^p \\ \sum_{i=1}^n X_i = X}} \sum_{i=1}^n F_i(X_i).$$

The following properties are well known (see e.g. [26]).

Lemma 2.2. (i) $\square_{i=1}^n F_i : L^p \to [-\infty, \infty]$ *is a convex function,*

(ii) $\text{dom} \, \square_{i=1}^n F_i = \sum_{i=1}^n \text{dom} \, F_i$,

(iii) $(\square_{i=1}^n F_i)^* = \sum_{i=1}^n F_i^*$,

(iv) $\operatorname{dom}(\square_{i=1}^n F_i)^* = \bigcap_{i=1}^n \operatorname{dom} F_i^*$.

Definition 2.3. *Let $X \in L^p$. An n-tuple $(X_1, X_2, \ldots, X_n) \in L^p \times \ldots \times L^p$ such that $\sum_{i=1}^n X_i = X$ is called an* allocation *of X. The convolution $\square_{i=1}^n F_i$ is said to be* exact at X *if there exists an allocation $(X_1, X_2, \ldots, X_n)$ of X such that $\square_{i=1}^n F_i(X) = \sum_{i=1}^n F_i(X_i)$. Such a minimizing allocation is called an* optimal allocation *of X. The convolution is said to be* exact *if it is exact at every point $X \in L^p$.*

Hence, the capital and risk allocation problem outlined in section 1 is equivalent to finding an optimal allocation for the convolution $\square_{i=1}^n \rho_i$ of n convex cash-invariant functions ρ_i.

Definition 2.4. *An allocation $(X_1, \ldots, X_n)$ of $X \in L^p$ is called* comonotone *if there exist increasing functions $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ such that $\sum_{i=1}^n f_i = \operatorname{Id}_{\mathbb{R}}$ and $X_i = f_i(X)$ for all i. These functions f_i are obviously 1-Lipschitz-continuous.*

The following existence theorem is the main result of this paper. The proof is given in section 6.

Theorem 2.5. *Let $\rho_1, \ldots, \rho_n : L^p \to (-\infty, \infty]$ be l.s.c. law-invariant convex cash-invariant functions. Then $\square_{i=1}^n \rho_i$ is a l.s.c. law-invariant convex cash-invariant function on L^p. Moreover, for every $X \in L^p$ there exists a comonotone allocation $(X_1, \ldots, X_n)$ such that*

$$\square_{i=1}^n \rho_i(X) = \sum_{i=1}^n \rho_i(X_i). \tag{2.4}$$

In other words, $\square_{i=1}^n \rho_i$ is exact, and amongst the optimal allocations of any $X \in L^p$ there is always a comonotone one.

Remark 2.6. The economic message of theorem 2.5 is that the capital and risk allocation problem (1.1) always admits a solution via contracts whose payoffs are defined as (increasing Lipschitz continuous) functions $f_i(X)$ of the aggregate risk X. We note that this extremely useful fact is often *assumed* in economic contract theory. Theorem 2.5 now sets this prevalent economic assumption on a sound mathematical basis. ‖

We note that the functions ρ_i in theorem 2.5 do not have to be monotone. In case at least one of them is monotone (i.e. a convex risk measure), we may draw the following stronger conclusion:

Corollary 2.7. *Let $\rho_1, \ldots, \rho_n : L^p \to (-\infty, \infty]$ be l.s.c. law-invariant convex cash-invariant functions, of which at least one is a convex risk measure. Then, $\square_{i=1}^n \rho_i$ is a l.s.c. law-invariant convex risk measure on L^p. Moreover, for every $X \in L^p$ there exists a comonotone optimal allocation.*

Proof. In view of theorem 2.5 it remains to prove that $\square_{i=1}^n \rho_i$ is monotone. But this follows immediately from lemma 2.2 and the fact that a l.s.c. proper convex function $F : L^p \to (-\infty, \infty]$ is monotone if and only if $\operatorname{dom} F^* \subset (L^p)^*_-$ (see e.g. [17] lemma 3.2). □

Next we apply theorem 2.5 to calculate optimal allocations for Average Value at Risks and entropic risk measures, respectively. These convolutions are discussed thoroughly in e.g. [4] or [22] on L^∞. In contrast, we provide our results on L^1.

Example 2.8. The Average Value at Risk (2.2) can be represented as

$$\text{AVaR}_\alpha(X) = \sup\left\{E_\mathbb{Q}[-X] \mid \mathbb{Q} \ll \mathbb{P},\ \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\alpha}\right\}. \qquad (2.5)$$

This representation is well known for $X \in L^\infty$ (see e.g. [19]) and is extended to L^1 in [18]. Now let $0 \leq \beta \leq \gamma \leq 1$, then

$$\text{AVaR}_\beta \,\square\, \text{AVaR}_\gamma = \text{AVaR}_\gamma. \qquad (2.6)$$

This is easily verified employing theorem 2.5, lemma 2.2, the Fenchel–Moreau theorem and the fact that $\text{dom}\,\text{AVaR}_\gamma^* \subset \text{dom}\,\text{AVaR}_\beta^*$, implied by (2.5). $\qquad \|$

Example 2.9. The entropic risk measure (2.3) can be represented as

$$\text{Entr}_\beta(X) = \sup_{\mathbb{Q} \in \mathcal{M}(\mathbb{P})} E_\mathbb{Q}[-X] - \frac{1}{\beta}H(\mathbb{Q} \mid \mathbb{P})$$

where $H(\mathbb{Q} \mid \mathbb{P}) = E_\mathbb{Q}[\log \frac{d\mathbb{Q}}{d\mathbb{P}}]$ denotes the relative entropy and $\mathcal{M}(\mathbb{P})$ is the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{Q} \ll \mathbb{P}$ and $d\mathbb{Q}/d\mathbb{P}$ is bounded. This representation is well known for $X \in L^\infty$ (see e.g. [19]) and is extended to L^1 in [18]. Now let $0 < \beta \leq \gamma$. Theorem 2.5 and lemma 2.2 justify the following dual approach, for any $X \in L^1$:

$$\begin{aligned}
\text{Entr}_\beta \,\square\, \text{Entr}_\gamma(X) &= \sup_{\mathbb{Q} \in \mathcal{M}(\mathbb{P})} E_\mathbb{Q}[-X] - \frac{1}{\beta}H(\mathbb{Q} \mid \mathbb{P}) - \frac{1}{\gamma}H(\mathbb{Q} \mid \mathbb{P}) \\
&= \sup_{\mathbb{Q} \in \mathcal{M}(\mathbb{P})} E_\mathbb{Q}[-X] - \frac{\beta + \gamma}{\beta\gamma}H(\mathbb{Q} \mid \mathbb{P}) \\
&= \text{Entr}_{\frac{\beta\gamma}{\beta+\gamma}}(X).
\end{aligned}$$

Now, in the search for comonotone optimal allocations, the following ansatz seems natural. We guess that for any $X \in L^1$ there must be an (obviously comonotone) optimal allocation amongst the allocations of type (aX, bX) where $a \in [0, 1]$ and $b := 1 - a$. If so, then

$$\frac{\beta + \gamma}{\beta\gamma}\log E[e^{-\frac{\beta\gamma}{\beta+\gamma}X}] = \frac{1}{\beta}\log E[e^{-\beta aX}] + \frac{1}{\gamma}\log E[e^{-\gamma bX}]$$

which is equivalent to

$$\log E[e^{-\frac{\beta\gamma}{\beta+\gamma}X}] = \frac{\gamma}{\beta + \gamma}\log E[e^{-\beta aX}] + \frac{\beta}{\beta + \gamma}\log E[e^{-\gamma bX}].$$

Clearly, $a = \frac{\gamma}{\beta+\gamma}$ and $b = \frac{\beta}{\beta+\gamma}$ satisfy this equation. Hence, $(\frac{\gamma}{\beta+\gamma}X, \frac{\beta}{\beta+\gamma}X)$ is a comonotone optimal allocation of X. $\qquad \|$

Remark 2.10. The l.s.c. requirement in theorem 2.5 cannot be dropped, as the following example shows. Let $\rho_1(X) = -E[X] + \delta(X^- \mid L^\infty)$, and $\rho_2 = \text{AVaR}_\alpha$ on L^1, for some $\alpha \in (0,1)$. Clearly, the acceptance set $\mathcal{A}_{\rho_1}$ of ρ_1 is not closed and thus ρ_1 is not l.s.c. on L^1. We claim that

$$\rho_1 \square \rho_2 = -E. \tag{2.7}$$

Indeed, on the one hand, we know that $\rho_1^* = \delta(\cdot \mid \{-1\})$ and that $\rho_2^*(-1) = 0$ (see (A.17) below). Hence, $(\rho_1 \square \rho_2)^* = \rho_1^* + \rho_2^* = \delta(\cdot \mid \{-1\})$ which implies $\rho_1 \square \rho_2 \geq -E$. On the other hand,

$$
\begin{aligned}
\rho_1 \square \rho_2(X) &= \inf_{X_1 + X_2 = X} E[-X_1] + \delta(X_1^- \mid L^\infty) + \text{AVaR}_\alpha(X_2) \\
&\leq \inf_{K \in \mathbb{N}} E[-X 1_{\{X > -K\}}] + \text{AVaR}_\alpha(X 1_{\{X \leq -K\}}) \\
&\leq E[-X] + \lim_{K \to \infty} \text{AVaR}_\alpha(X 1_{\{X \leq -K\}}) = E[-X]
\end{aligned}
$$

because AVaR_α is continuous and $X 1_{\{X \leq -K\}} \to 0$ in L^1 for $K \to \infty$. This proves (2.7).

Now, choose any $X \in L^1$ being unbounded from below. Suppose there is an optimal allocation (X_1, X_2) of X. Then X_1 must be bounded and X_2 unbounded from below, respectively. In view of lemma 2.11 below, we thus have $\text{AVaR}_\alpha(X_2) > E[-X_2]$, and hence

$$
\begin{aligned}
\rho_1 \square \rho_2(X) &= \rho_1(X_1) + \rho_2(X_2) \\
&= E[-X_1] + \delta(X_1^- \mid L^\infty) + \text{AVaR}_\alpha(X_2) \\
&> E[-X_1] + E[-X_2] = E[-X],
\end{aligned}
$$

which contradicts (2.7). Hence, there exists no optimal allocation of X. ‖

Lemma 2.11. *Let $0 \leq \beta < \gamma \leq 1$. Then*

$$\text{AVaR}_\beta(X) \geq \text{AVaR}_\gamma(X),$$

and equality holds if and only if $X \geq c$ a.s. and $\mathbb{P}[X = c] \geq \gamma$ for some constant $c \in \mathbb{R}$. In particular, $\text{AVaR}_\beta(X) = E[-X]$ if and only if X is constant.

Proof. The case $\beta = 0$ is obvious. Suppose $\beta > 0$. Since q_X is increasing, we obviously have

$$(\gamma - \beta) \int_0^\beta q_X(s)\,ds \leq \beta \int_\beta^\gamma q_X(s)\,ds,$$

with equality if and only if $q_X(s) = q_X(\gamma)$ for all $s \leq \gamma$. This proves the claim. □

Remark 2.12. Also the law-invariance requirement in theorem 2.5 cannot be dropped: let $Z \geq 0$ non-constant with $E[Z] = 1$. Then $\rho_1 = -E$ and $\rho_2 = E[-Z\cdot]$ are convex risk measures on L^∞ and ρ_2 is not law-invariant. Thus theorem 2.5 does not apply. Indeed, the convolution $\rho_1 \square \rho_2 \equiv -\infty$ is not exact.‖

3 Uniqueness of Optimal Allocations

Let $\rho_1, \ldots, \rho_n : L^p \to (-\infty, \infty]$ be convex cash-invariant functions. Due to cash-invariance of ρ_i, uniqueness of an optimal allocation can only hold up to *rebalancing the cash*. That is, $(X_1, \ldots, X_n)$ is an optimal allocation of X if and only if $(X_1 + c_1, \ldots, X_n + c_n)$ is so, for all cash positions $c_i \in \mathbb{R}$ with $\sum_{i=1}^n c_i = 0$.

The following sufficient condition for uniqueness is straightforward.

Proposition 3.1. *Suppose $\rho_1, \ldots, \rho_{n-1}$ are strictly convex in the sense that*

$$\rho_i(\lambda X + (1 - \lambda)Y) < \lambda \rho_i(X) + (1 - \lambda)\rho_i(Y) \quad \text{for all } \lambda \in (0, 1), \qquad (3.8)$$

for all $X, Y \in \operatorname{dom} \rho_i$ with $X - Y \notin \mathbb{R}$. Then any optimal allocation of $X \in L^p$ with $\square_{i=1}^n \rho_i(X) < \infty$ is unique up to rebalancing the cash.

Proof. Let $X \in L^p$ with $\square_{i=1}^n \rho_i(X) < \infty$. We argue by contradiction and suppose $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ are optimal allocations of X with $X_j - Y_j \notin \mathbb{R}$ for some $1 \leq j \leq n - 1$. Then, for any $\lambda \in (0, 1)$, $Z_i = \lambda X_i + (1 - \lambda)Y_i$ is an allocation of X with

$$\sum_{i=1}^n \rho_i(Z_i) < \lambda \sum_{i=1}^n \rho_i(X_i) + (1 - \lambda) \sum_{i=1}^n \rho_i(Y_i) = \square_{i=1}^n \rho_i(X).$$

But this contradicts the optimality of X_i, whence the claim. $\qquad \square$

For instance, the optimal allocation for the entropic risk measure in example 2.9 is unique up to rebalancing the cash. More recent examples of strictly convex risk measures can be found in [10].

Without the strict convexity assumption in proposition 3.1, uniqueness does not hold in general. A trivial example is

$$(-E) \square (-E) = -E.$$

In this case, all allocations of any $X \in L^1$ are optimal allocations of X.

Moreover, the following example shows that uniqueness may even fail in the class of comonotone allocations without the strict convexity assumption in proposition 3.1.

Example 3.2. Let $0 \leq \beta \leq \gamma < 1$. In view of (2.5), there exists some $A \in \mathcal{F}$ with $\mathbb{P}[A] > 0$ and some $\mathbb{Q} \ll \mathbb{P}$ with $d\mathbb{Q}/d\mathbb{P} \leq 1/\gamma \leq 1/\beta$ and $\mathbb{Q}[A] = 0$. Consequently,

$$0 = \operatorname{AVaR}_\beta(1_A) = \operatorname{AVaR}_\gamma(1_A) = \operatorname{AVaR}_\beta \square \operatorname{AVaR}_\gamma(1_A).$$

Hence, both $(1_A, 0)$ and $(0, 1_A)$ are comonotone optimal allocations of 1_A. $\qquad \|$

On the other hand, the strict convexity assumption in proposition 3.1 is not necessary for uniqueness up to rebalancing the cash. This is shown by the following example.

Example 3.3. Let $\beta \in (0, 1)$. We know that $\operatorname{AVaR}_\beta \square -E = -E$ (example 2.8). Suppose $(Y, X - Y)$ is an optimal allocation of X. This implies $\operatorname{AVaR}_\beta(Y) + E[-(X - Y)] = E[-X]$, that is, $\operatorname{AVaR}_\beta(Y) = E[-Y]$. In view of lemma 2.11 we conclude that Y must be constant, i.e. a cash position. $\qquad \|$

4 Problem Reduction

First note that it is enough to prove theorem 2.5 for $n = 2$, the rest follows by induction. Indeed, if $\rho_1 \square \cdots \square \rho_{n-1} > -\infty$ then associativity holds:

$$\square_{i=1}^n \rho_i = \left(\square_{i=1}^{n-1} \rho_i \right) \square \rho_n.$$

For the sake of simplicity, we will further restrict our studies to the case $p = 1$. By nature of the arguments presented in the proof of theorem 2.5 (section 6), it will become clear that they all literally carry over to L^p, simply by replacing L^1 with L^p and choosing the appropriate dual. However, in what follows, we give another justification for the retreat to L^1 by proving that the assertions of theorem 2.5 for the L^p-cases can be derived as a corollary from knowing it for the case of L^1. The main ingredient to this is the following result:

Theorem 4.1. *Let $\rho : L^p \to (-\infty, \infty]$ be a l.s.c. law-invariant convex cash-invariant function. Then, there is a unique l.s.c. law-invariant convex cash-invariant function $\widetilde{\rho} : L^1 \to (-\infty, \infty]$ such that $\rho = \widetilde{\rho}|_{L^p}$.*

Proof. see [18]. □

Let
$$\mathbb{A} := \{(f, g) \mid f, g : \mathbb{R} \to \mathbb{R} \text{ are increasing}, \ f + g = \mathrm{Id}_\mathbb{R}\}.$$

Clearly, if $(f, g) \in \mathbb{A}$, then both f and g are 1-Lipschitz-continuous. Hence, $|f(X)| \leq |X| + |f(0)|$ and $|g(X)| \leq |X| + |g(0)|$, implying that

$$\text{if } X \in L^p \text{ then } (f(X), g(X)) \in L^p \times L^p. \tag{4.9}$$

Thus, for any $X \in L^p$ the set $\{(f(X), g(X)) \mid (f, g) \in \mathbb{A}\}$ is the set of all comonotone 2-dimensional allocations of X.

Corollary 4.2. *Let $\rho_1, \rho_2 : L^p \to (-\infty, \infty]$ be two l.s.c. law-invariant convex cash-invariant functions, and let $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ be the unique l.s.c. law-invariant convex cash-invariant functions on L^1 such that $\rho_i = \widetilde{\rho}_i|_{L^p}$, $i = 1, 2$. Suppose the assertions of theorem 2.5 hold for $\widetilde{\rho}_1 \square \widetilde{\rho}_2$. Then $\rho_1 \square \rho_2 = \widetilde{\rho}_1 \square \widetilde{\rho}_2|_{L^p}$. In particular, the assertions of theorem 2.5 are true for $\rho_1 \square \rho_2$ too.*

Proof. By assumption, for any $X \in L^p \subset L^1$ there is a comonotone allocation $(f(X), g(X))$ such that $\widetilde{\rho}_1 \square \widetilde{\rho}_2(X) = \widetilde{\rho}_1(f(X)) + \widetilde{\rho}_2(g(X))$. Clearly, $\widetilde{\rho}_1 \square \widetilde{\rho}_2(X) \leq \rho_1 \square \rho_2(X)$, and since $(f(X), g(X)) \in L^p \times L^p$ by (4.9), we deduce that

$$\widetilde{\rho}_1 \square \widetilde{\rho}_2(X) = \rho_1(f(X)) + \rho_2(g(X)) = \rho_1 \square \rho_2(X).$$

Hence, firstly, $\rho_1 \square \rho_2$ is simply the restriction of $\widetilde{\rho}_1 \square \widetilde{\rho}_2$ to L^p and thus a l.s.c. (w.r.t. $\|\cdot\|_p$) law-invariant convex cash-invariant function. (The l.s.c. stems from the fact that $\|\cdot\|_p$-convergence implies $\|\cdot\|_1$-convergence.) Secondly, $\rho_1 \square \rho_2$ is exact and there is always a comonotone optimal allocation. □

5 Comonotone Concave Order Improvement

We denote by $\succeq_c$ the concave order on L^1, that is, $X \succeq_c Y$ if and only if $E[u(X)] \geq E[u(Y)]$ for all concave functions $u : \mathbb{R} \to \mathbb{R}$. Clearly, since $\mathrm{Id}_\mathbb{R}$ and $-\mathrm{Id}_\mathbb{R}$ are concave functions, $X \succeq_c Y$ implies $E[X] = E[Y]$. Moreover,

$$X \succeq_c Y \quad \Leftrightarrow \quad E[X] = E[Y] \text{ and } E[(X - c)^+] \leq E[(Y - c)^+] \, \forall c \in \mathbb{R}. \quad (5.10)$$

For a proof of (5.10), we refer to corollary 2.62 in [19].

Proposition 5.1 below will turn out to be the basis of the proof of theorem 2.5. It is based upon the results of Landsberger and Meilijson [25], and states that every allocation is dominated in concave order by a comonotone allocation. The importance of this results becomes clear by (A.14) where we establish that any l.s.c. law-invariant convex function is monotone w.r.t. the $\succeq_c$-order.

Proposition 5.1. *(see proposition 1 in [25]) For any allocation (Y, Z) of $X \in L^1$, there is $(f, g) \in \mathbb{A}$ such that $f(X) \succeq_c Y$ and $g(X) \succeq_c Z$.*

Unfortunately, Landsberger and Meilijson [25] only proved this result for random variables X supported by a finite set. For sake of completeness, we thus give a full proof here.

Proof. We divide the proof into three steps.

Step 1: We start out as in [25] by noticing that Jensen's inequality implies that $(E[Y \mid X], E[Z \mid X])$ is an allocation of X which is at least as good as (Y, Z), meaning that $E[Y \mid X] \succeq_c Y$ and $E[Z \mid X] \succeq_c Z$. Let $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ be measurable functions such that $h_1(X) = E[Y \mid X]$, $h_2(X) = E[Z \mid X]$. Clearly, we may assume that $h_1 + h_2 = \mathrm{Id}_\mathbb{R}$. If h_1 and h_2 are increasing, we are done, if not, we go on improving this allocation. However, we have now established that during the remainder of this proof we may restrict ourselves to improve allocations (Y, Z) of type $Y = h_1(X)$ and $Z = h_2(X)$ for some measurable functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ such that $h_1 + h_2 = \mathrm{Id}_\mathbb{R}$.

Step 2: Suppose X is a simple random variable, i.e. $X = \sum_{i=1}^n x_i 1_{A_i}$ for a partition $A_1, \ldots, A_n$ of Ω and real numbers x_i such that $x_i \neq x_j$ for $i \neq j$. Let $y_i := h_1(x_i)$ and $z_i := h_2(x_i)$. Then $h_1(X) = \sum_{i=1}^n y_i 1_{A_i}$ and $h_2(X) = \sum_{i=1}^n z_i 1_{A_i}$. We set $x := (x_1, \ldots, x_n)$, $y := (y_1, \ldots, y_n)$, $z := (z_1, \ldots, z_n)$ and $p_k := \mathbb{P}(A_k)$, $k = 1, \ldots, n$. Let π be a permutation of $\{1, \ldots, n\}$ such that $x_\pi := (x_{\pi(1)}, \ldots, x_{\pi(n)}) \in \mathcal{D} := \{\tilde{x} \in \mathbb{R}^n \mid \tilde{x}_1 \leq \tilde{x}_2 \leq \ldots \leq \tilde{x}_n\}$. Observe that $(h_1(X), h_2(X))$ is comonotone if and only if $y_\pi, z_\pi \in \mathcal{D}$. For sake of brevity we may and will assume w.l.o.g. that $x \in \mathcal{D}$ already. Supposing that (y, z) is not comonotone, i.e. $y \notin \mathcal{D}$ or $z \notin \mathcal{D}$ or both, the following algorithm by M. Landsberger and I. Meilijson transfers (y, z) into a comonotone allocation:

Since (y, z) is not comonotone, there must exist an i such that $y_1 \leq \ldots \leq y_i$, $z_1 \leq \ldots \leq z_i$ but either $y_{i+1} < y_i$ or $z_{i+1} < z_i$. W.l.o.g. let us assume that $z_{i+1} < z_i$. Then there is a smallest j such that $z_{i+1} < z_j$. For $k = j, \ldots, i$ we set

$$y_k^{new} = y_k + \frac{p_{i+1}}{\sum_{l=j}^{i+1} p_l}(z_j - z_{i+1}) \quad \text{and} \quad z_k^{new} = z_k - \frac{p_{i+1}}{\sum_{l=j}^{i+1} p_l}(z_j - z_{i+1})$$

whereas

$$y_{i+1}^{new} = y_{i+1} - \frac{\sum_{l=j}^{i} p_l}{\sum_{l=j}^{i+1} p_l}(z_j - z_{i+1}) \quad \text{and} \quad z_{i+1}^{new} = z_{i+1} + \frac{\sum_{l=j}^{i} p_l}{\sum_{l=j}^{i+1} p_l}(z_j - z_{i+1}).$$

The other coordinates of y and z are left unchanged. Finally, set $y := y^{new}$ and $z := z^{new}$ and repeat the procedure in case the output is not comonotone.

Let $(Y^{new}, Z^{new}) := (\sum_{i=1}^{n} y_i^{new} 1_{A_i}, \sum_{i=1}^{n} z_i^{new} 1_{A_i})$. Firstly, (Y^{new}, Z^{new}) is obviously an allocation of X, secondly, we claim that $Y^{new} \succeq_c Y$ and $Z^{new} \succeq_c Z$, i.e. each cycle of the algorithm improves the allocation, and finally, it is easily verified that the algorithm returns a comonotone allocation in at most $n(n-1)/2$ cycles (observe that $z_j^{new} = z_{i+1}^{new}$). In order to show that $Y^{new} \succeq_c Y$ and $Z^{new} \succeq_c Z$, let $u : \mathbb{R} \to \mathbb{R}$ be any concave function. Introducing the abbreviations

$$\alpha := \frac{p_{i+1}}{\sum_{l=j}^{i+1} p_l} \in (0,1) \quad \text{and} \quad \lambda_k := \frac{z_j - z_{i+1}}{z_k - z_{i+1}} \in (0,1]$$

and recalling that concavity is equivalent to

$$\forall a < b < c : \quad \frac{u(b) - u(a)}{b - a} \geq \frac{u(c) - u(a)}{c - a} \geq \frac{u(c) - u(b)}{c - b}, \qquad (5.11)$$

we compute:

$$\sum_{k=j}^{i+1} u(z_k^{new}) p_k = \sum_{k=j}^{i} u((1 - \alpha\lambda_k)z_k + \alpha\lambda_k z_{i+1}) p_k + u((1 - \alpha)z_j + \alpha z_{i+1}) p_{i+1}$$

$$\geq \sum_{k=j}^{i} \left[(1 - \alpha\lambda_k)u(z_k) + \alpha\lambda_k u(z_{i+1})\right] p_k$$

$$+ \left[(1 - \alpha)u(z_j) + \alpha u(z_{i+1})\right] p_{i+1}$$

$$= \sum_{k=j}^{i+1} u(z_k) p_k + (1 - \alpha)(u(z_j) - u(z_{i+1})) p_{i+1}$$

$$- \alpha \sum_{k=j}^{i} \lambda_k (u(z_k) - u(z_{i+1})) p_k$$

$$\overset{(5.11)}{\geq} \sum_{k=j}^{i+1} u(z_k) p_k,$$

because $\lambda_k(u(z_k) - u(z_{i+1})) \leq u(z_j) - u(z_{i+1})$ by inequality (5.11). A similar computation for Y^{new} shows that $Y^{new} \succeq_c Y$ and $Z^{new} \succeq_c Z$.

Step 3: Let X be any integrable random variable. Recalling the usual monotone approximation from Lebesgue integration theory, let $(Y_n)_{n\in\mathbb{N}}$ and $(Z_n)_{n\in\mathbb{N}}$ be sequences of simple random variables converging $\mathbb{P}$-a.s. and in L^1 to

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Y and Z respectively such that $|Y_n| \leq |Y|$ and $|Z_n| \leq |Z|$ for all $n \in \mathbb{N}$. Then $X_n := Y_n + Z_n$ converges to X $\mathbb{P}$-a.s. and in L^1. By step 2, for each $n \in \mathbb{N}$, there exists a comonotone improvement $(f_n(X_n), g_n(X_n))$ of (Y_n, Z_n). Choose $N \in \mathbb{N}$ such that $\|Y_n\|_1 \leq \|Y\|_1 + 1$, $\|Z_n\|_1 \leq \|Z\|_1 + 1$, and $\|X_n\|_1 \leq \|X\|_1 + 1$ for all $n \geq N$. Since all f_n (and g_n) are 1-Lipschitz-continuous, we have that $|f_n(0)| \leq |X_n| + |f_n(X_n)|$. Taking expectations on both sides yields

$$|f_n(0)| \leq E[|X_n|] + E[|f_n(X_n)|] \leq E[|X_n|] + E[|Y_n|],$$

because $f_n(X_n) \succeq_c Y_n$ and $x \mapsto -|x|$ is concave. Hence, if $n \geq N$, we get $|f_n(0)| \leq \|X\|_1 + \|Y\|_1 + 2 =: K_1$ and similarly $|g_n(0)| \leq \|X\|_1 + \|Z\|_1 + 2 =: K_2$, and thus $f_n(0), g_n(0) \in [-K, K]$ for $K := \max\{K_1, K_2\}$. Therefore, by lemma B.1, there is a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ and a 1-Lipschitz-continuous increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f(a) = \lim_{k \to \infty} f_{n_k}(a)$, $a \in \mathbb{R}$. Now it is easily verified that $(g_{n_k})_{k \in \mathbb{N}}$ converges pointwise to the 1-Lipschitz-continuous increasing function $g := \mathrm{Id}_\mathbb{R} - f$. Hence, the sequence $f_{n_k}(X_{n_k})$ converges $\mathbb{P}$-a.s. to $f(X)$, and $g_{n_k}(X_{n_k}) = X_{n_k} - f_{n_k}(X_{n_k})$ converges $\mathbb{P}$-a.s. to $g(X)$. Since $|f_{n_k}(X_{n_k})| \leq |X_{n_k}| + K \leq |Y| + |Z| + K$ and $|g_{n_k}(X_{n_k})| \leq |Y| + |Z| + K$ for large enough $k \in \mathbb{N}$, we can apply the dominated convergence theorem which yields $f(X), g(X) \in L^1$ and $\|f(X) - f_{n_k}(X_{n_k})\|_1 \to 0$, $\|g(X) - g_{n_k}(X_{n_k})\|_1 \to 0$ for $k \to \infty$. Moreover, in view of (5.10), we have that

$$E[f(X)] = \lim_{k \to \infty} E[f_{n_k}(X_{n_k})] = \lim_{k \to \infty} E[Y_{n_k}] = E[Y],$$

and for all $c \in \mathbb{R}$:

$$
\begin{aligned}
E[(f(X) - c)^+] &= \lim_{k \to \infty} E[(f_{n_k}(X_{n_k}) - c)^+] \\
&\leq \lim_{k \to \infty} E[(Y_{n_k} - c)^+] = E[(Y - c)^+],
\end{aligned}
$$

and similarly for g. Hence, $(f(X), g(X))$ is a comonotone allocation of X satisfying $f(X) \succeq_c Y$ and $g(X) \succeq_c Z$ according to (5.10). $\qquad\square$

6 Proof of Theorem 2.5

In view of section 4 we only have to prove theorem 2.5 for $n = 2$ and $p = 1$. To this end, let $\rho_1, \rho_2 : L^1 \to (-\infty, \infty]$ be l.s.c. law-invariant convex cash-invariant functions. We divide the proof into four steps.

Step 1: $\rho_1 \square \rho_2$ is proper, convex, and cash-invariant.

Proof. It is easily verified that the convolution preserves the convexity of ρ_1 and ρ_2. By (A.16) we have that $\rho_i(X) \geq -E[X] + \rho_i(0)$ for all $X \in L^1$ and $i = 1, 2$. Hence,

$$
\begin{aligned}
\rho_1 \square \rho_2(X) &= \inf_{X_1 + X_2 = X} \rho_1(X_1) + \rho_2(X_2) \\
&\geq \inf_{X_1 + X_2 = X} -E[X_1] - E[X_2] + \rho_1(0) + \rho_2(0) \\
&= -E[X] + \rho_1(0) + \rho_2(0),
\end{aligned}
$$

so $\rho_1 \square \rho_2$ is proper and $\rho_1 \square \rho_2(0) = \rho_1(0) + \rho_2(0) < \infty$. Furthermore, for all $r \in \mathbb{R}$ we get

$$\rho_1 \square \rho_2(X + r) = \inf_{Y \in L^1} \rho_1(X + r - Y) + \rho(Y) = \rho_1 \square \rho_2(X) - r$$

due to the cash-invariance of ρ_1. $\qquad\square$

Step 2: $\rho_1 \square \rho_2(X) = \inf_{(f,g) \in \mathbb{A}} \rho_1(f(X)) + \rho_2(g(X)), \quad X \in L^1.$

Proof. This is an immediate consequence of proposition 5.1 and (A.14). $\qquad\square$

Step 3: $\rho_1 \square \rho_2$ is exact, and for each $X \in L^1$ there exists a comonotone optimal allocation.

Proof. Suppose $X \in L^1$ is such that $\rho_1 \square \rho_2(X) = \infty$. Then every allocation $(f(X), g(X))$, $(f, g) \in \mathbb{A}$, is optimal.

Now let $X \in \mathrm{dom}\, \rho_1 \square \rho_2$ and choose a sequence $(f_n, g_n) \in \mathbb{A}$, $n \in \mathbb{N}$, such that $\rho_1 \square \rho_2(X) = \lim_{n \to \infty} \rho_1(f_n(X)) + \rho_2(g_n(X))$. By cash-invariance we may assume that $f_n(0) = g_n(0) = 0$ for all $n \in \mathbb{N}$. Hence, by lemma B.1, there is a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ and a 1-Lipschitz-continuous and increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f(a) = \lim_{k \to \infty} f_{n_k}(a)$ for all $a \in \mathbb{R}$. Clearly, the sequence $f_{n_k}(X)$ converges $\mathbb{P}$-a.s. to $f(X)$ and $g_{n_k}(X) = X - f_{n_k}(X)$ converges $\mathbb{P}$-a.s. to $g(X)$ where $g := \mathrm{Id}_{\mathbb{R}} - f$ is a 1-Lipschitz-continuous increasing function. Since $|f_{n_k}(X)| \leq |X|$ and $|g_{n_k}(X)| \leq |X|$ for all $k \in \mathbb{N}$, we may apply the dominated convergence theorem which yields $f(X), g(X) \in L^1$ and $\|f(X) - f_{n_k}(X)\|_1 \to 0$, $\|g(X) - g_{n_k}(X)\|_1 \to 0$ for $k \to \infty$. On the one hand, by l.s.c., we have

$$
\begin{aligned}
\rho_1 \square \rho_2(X) &= \lim_{k \to \infty} \rho_1(f_{n_k}(X)) + \rho_2(g_{n_k}(X)) \\
&\geq \liminf_{k \to \infty} \rho_1(f_{n_k}(X)) + \liminf_{k \to \infty} \rho_2(g_{n_k}(X)) \\
&\overset{l.s.c.}{\geq} \rho_1(f(X)) + \rho_2(g(X)).
\end{aligned}
$$

On the other hand, by very definition of the convolution, we have $\rho_1 \square \rho_2(X) \leq \rho_1(f(X)) + \rho_2(g(X))$. Consequently, the comonotone allocation $(f(X), g(X))$ of X is optimal. $\qquad\square$

Step 4: $\rho_1 \square \rho_2$ is l.s.c. and law-invariant.

Proof. We claim that $\mathcal{A}_{\rho_1 \square \rho_2}$ is closed. To this end, let $(X_n)_{n \in \mathbb{N}} \subset \mathcal{A}_{\rho_1 \square \rho_2}$ be a sequence converging to some X in L^1. According to step 3 there are $(f_n, g_n) \in \mathbb{A}$, $n \in \mathbb{N}$, such that $0 \geq \rho_1 \square \rho_2(X_n) = \rho_1(f_n(X_n)) + \rho_2(g_n(X_n))$. By cash-invariance we may assume that $f_n(0) = g_n(0) = 0$ for all $n \in \mathbb{N}$. Similar to step 3, employing lemma B.1, we find a subsequence $(f_{n_k}, g_{n_k})_{k \in \mathbb{N}}$

of $(f_n, g_n)_{n \in \mathbb{N}}$ and $(f, g) \in \mathbb{A}$ such that $f_{n_k}(X_{n_k})$ converges to $f(X)$ in L^1 and $g_{n_k}(X_{n_k})$ converges to $g(X)$ in L^1. By l.s.c. of ρ_1 and ρ_2 we get

$$
\begin{aligned}
\rho_1 \square \rho_2(X) &\leq \rho_1(f(X)) + \rho_2(g(X)) \\
&\leq \liminf_{k \to \infty} \rho_1(f_{n_k}(X_{n_k})) + \liminf_{k \to \infty} \rho_2(g_{n_k}(X_{n_k})) \\
&\leq \liminf_{k \to \infty} \rho_1(f_{n_k}(X_{n_k})) + \rho_2(g_{n_k}(X_{n_k})) \leq 0,
\end{aligned}
$$

and thus $X \in \mathcal{A}_{\rho_1 \square \rho_2}$. Hence, $\mathcal{A}_{\rho_1 \square \rho_2}$ is closed, i.e. $\rho_1 \square \rho_2$ is l.s.c.. The law-invariance of $\rho_1 \square \rho_2$ follows from lemma 2.2 and the fact that a l.s.c. convex function on L^1 is law-invariant if and only if its dual is (see (A.13)). $\qquad \square$

7   Optimal Risk Sharing under Constraints

This application is motivated and explained in [14]. Two agents with initial endowments X_1 and X_2 in L^p, assess their individual risk by means of l.s.c. law-invariant convex risk measures ρ_1 and ρ_2 on L^p, respectively. In order to minimize total and individual risk, they redistribute the aggregate endowment $X = X_1 + X_2$ amongst themselves. As is often the case in practice, this redistribution procedure might be subject to some restrictions in the sense that not every risk sharing of X is admissible. We formalize this by defining the set of admissible risk sharings of X as

$$
A_X := \{ (Y_1, Y_2) \in M_1 \times M_2 \mid Y_1 + Y_2 \leq X \}
$$

where $M_i \subset L^p$ are closed convex law-invariant cash-invariant (that is, $Y \in M_i$ implies $Y + a \in M_i$ for all $a \in \mathbb{R}$) sets such that $X_i \in M_i$, $i = 1, 2$. Note that we allow for "free disposal", i.e. $X - Y_1 - Y_2 \geq 0$ for all $(Y_1, Y_2) \in A_X$. The optimal risk sharing under constraints problem becomes

$$
\inf_{(Y_1, Y_2) \in A_X} \rho_1(Y_1) + \rho_2(Y_2). \tag{7.12}
$$

In order to solve (7.12), denote $\rho_i^{M_i} := \rho_i + \delta(\cdot \mid M_i)$, $i = 1, 2$. Then

$$
\begin{aligned}
(7.12) &= \inf_{Y_1, Y_2 \in L^p, \, Y_1 + Y_2 \leq X} \rho_1^{M_1}(Y_1) + \rho_2^{M_2}(Y_2) \\
&= \rho_1^{M_1} \square \rho_2^{M_2} \square \delta(\cdot \mid L_+^p)(X) \\
&= \rho_1^{M_1} \square \rho_2^{M_2} \square - \operatorname{essinf}(X).
\end{aligned}
$$

Note that $\delta(\cdot \mid M_i)$ is proper, l.s.c., law-invariant, and convex. By (A.15) we know that $\delta(E[Y] \mid M_i) \leq \delta(Y \mid M_i)$ for all $Y \in L^p$. Hence $Y \in M_i$ implies $E[Y] \in M_i$, and thus $\mathbb{R} \subset M_i$ by cash-invariance. We conclude that $\rho_1^{M_1}$ and $\rho_2^{M_2}$ are l.s.c. law-invariant convex cash-invariant functions. Since $-\operatorname{essinf}$ is a l.s.c. law-invariant coherent risk measure, we know by corollary 2.7 that $\rho_1^{M_1} \square \rho_2^{M_2} \square -$ essinf is a l.s.c. law-invariant convex risk measure, and that this convolution admits a comonotone optimal allocation (Y_1, Y_2, Y_3) of X. If $\rho_1^{M_1}(Y_1) + \rho_2^{M_1}(Y_2) -$

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essinf$(Y_3) = \infty$, then any admissible risk sharing of X is optimal. Otherwise, if $\rho_1^{M_1}(Y_1) + \rho_2^{M_1}(Y_2) - \text{essinf}(Y_3) < \infty$, then we have that $-\text{essinf}(Y_3) < \infty$, and thus $(Y_1 + \text{essinf}(Y_3), Y_2)$ is a solution of the optimization problem (7.12). We thus have established a substantial improvement of corollary 3.8 in [14]. Note also that $(Y_1 + \text{essinf}(Y_3), Y_2) = (f(X), g(X))$ for some increasing functions $f, g : \mathbb{R} \to \mathbb{R}$.

A   Law-invariant Convex Functions on $\mathbf{L^p}$

The proof of theorem 2.5 draws heavily on the following properties, which are proved [18]. Let $F : L^p \to (-\infty, \infty]$ be a proper l.s.c. convex function, then the following conditions are equivalent:

$$F \text{ is law-invariant} \Leftrightarrow F^* \text{ is law-invariant} \tag{A.13}$$

$$\Leftrightarrow X \succeq_c Y \text{ implies } F(X) \leq F(Y). \tag{A.14}$$

Consequently, by Jensen's inequality, if F is law-invariant, then

$$F(E[X \mid \mathcal{G}]) \leq F(X) \text{ for all sub-}\sigma\text{-algebras } \mathcal{G} \subset \mathcal{F}. \tag{A.15}$$

Now let $\rho : L^p \to (-\infty, \infty]$ be a l.s.c. law-invariant convex cash-invariant function. Then cash-invariance and (A.15) imply that

$$\rho(X) \geq -E[X] + \rho(0). \tag{A.16}$$

Clearly,

$$\rho^*(-1) = \sup_{X \in L^p} E[-X] - \rho(X) \geq -\rho(0).$$

However, if $\rho^*(-1) > -\rho(0)$, there must be some $Y \in L^p$ such that $E[-Y] - \rho(Y) > -\rho(0)$. But this contradicts (A.16). We thus have shown

$$\rho^*(-1) = -\rho(0). \tag{A.17}$$

B   An Arzela-Ascoli Type Argument

The following lemma is needed for the proofs in sections 5 and 6.

Lemma B.1. *Let $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of increasing 1-Lipschitz-continuous functions such that $f_n(0) \in [-K, K]$ for all $n \in \mathbb{N}$ where $K \geq 0$ is a constant. Then there is a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ and an increasing 1-Lipschitz-continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{k \to \infty} f_{n_k}(x) = f(x)$ for all $x \in \mathbb{R}$.*

Proof. The Lipschitz-continuity guarantees that $f_n(x) \in [-K, K + x]$ if $x \geq 0$ and $f_n(x) \in [-K + x, K]$ if $x \leq 0$. Hence, by a procedure well-known from the standard proof of the Arzela-Ascoli theorem, we are able to extract a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that $\lim_{k \to \infty} f_{n_k}(q)$ exists for all $q \in \mathbb{Q}$. In

fact, we can easily show that the sequences $(f_{n_k}(x))_{k\in\mathbb{N}}$ must converge for all $x \in \mathbb{R}$. To this end, let $\epsilon > 0$ be arbitrary and choose $q \in \mathbb{Q}$ and $N_0 \in \mathbb{N}$ such that $|q - x| < \epsilon/3$ and $|f_{n_k}(q) - f_{n_l}(q)| < \epsilon/3$ for all $k, l \geq N_0$. Then for all $k, l \geq N_0$:

$$
\begin{aligned}
|f_{n_k}(x) - f_{n_l}(x)| &\leq |f_{n_k}(x) - f_{n_k}(q)| + |f_{n_k}(q) - f_{n_l}(q)| + \\
&\qquad\qquad\qquad\qquad + |f_{n_l}(q) - f_{n_l}(x)| \\
&\leq 2|x - q| + |f_{n_k}(q) - f_{n_l}(q)| < \epsilon ,
\end{aligned}
$$

in which we did apply the Lipschitz-continuity twice. Now it is easily verified that $f(x) := \lim_{k\to\infty} f_{n_k}(x)$, $x \in \mathbb{R}$, is a 1-Lipschitz-continuous increasing function. $\qquad\square$

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